Formative assessment #4

A. Find the general solutions of the following linear homogeneous ODEs of 2^{nd}-order

1) $y'' + y' - 12y = 0$

Characteristic equation $\quad M_2(\lambda) = \lambda^2 + \lambda - 12 = 0$

Roots $\quad \lambda = \dfrac{-1 \pm \sqrt{1 + 4 \cdot 12}}{2} = \dfrac{-1 \pm \sqrt{49}}{2} = \dfrac{-1 \pm 7}{2} = \begin{cases} \dfrac{-1+7}{2} = \dfrac{6}{2} = 3 \\ \dfrac{-1-7}{2} = -4 \end{cases}$

Two real roots of the characteristic equation

General Solution of the ODE:

$y(x) = C_1 e^{3x} + C_2 e^{-4x}$ where $C_1, C_2 \in \mathbb{R}$ are arbitrary constants.

3) $y'' + 2y' + 17y = 0$

Characteristic equation $\quad M_2(\lambda) = \lambda^2 + 2\lambda + 17 = 0$

Roots

$\lambda = \dfrac{-2 \pm \sqrt{4 - 4 \cdot 17}}{2} = \dfrac{-2 \pm \sqrt{4-68}}{2} = \dfrac{-2 \pm \sqrt{64}\, i}{2} = \dfrac{-2 \pm 8i}{2}$

$\lambda_1 = \dfrac{-2-8i}{2} = -1 - 4i \qquad \lambda_2 = \dfrac{-2+8i}{2} = -1 + 4i$

Two complex conjugate roots $\quad \lambda_1 = \alpha - i\beta \quad \lambda_2 = \alpha + i\beta$

$\alpha = -1 \quad \beta = 4$

General solution

$$y(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} = C_1 e^{(\alpha - i\beta)x} + C_2 e^{(\alpha + i\beta)x}$$

or alternatively

$$y(x) = e^{\alpha x}\left(A \cos \beta x + B \sin \beta x\right)$$

In our case

$$y(x) = C_1 e^{(-1-4i)x} + C_2 e^{(-1+4i)x}$$

where $C_1, C_2 \in \mathbb{R}$ are arbitrary constants

or

$$y(x) = e^{-x}\left(A \cos 4x + B \sin x\right)$$

where $A, B \in \mathbb{R}$ are arbitrary constants.

B. Solve the IVP.

$$10y'' - y' - 3y = 0 \quad , \quad y(0) = 1 \quad y'(0) = 0$$

Characteristic equation $\quad M_2(\lambda) = 10\lambda^2 - \lambda - 3 = 0$

Roots

$$\lambda = \frac{1 \pm \sqrt{1 + 4 \cdot 3 \cdot 10}}{20} = \frac{1 \pm \sqrt{121}}{20} = \frac{1 \pm 11}{20} = \begin{cases} -\frac{10}{20} = -\frac{1}{2} \\ \frac{12}{20} = \frac{3}{5} \end{cases}$$

General solution

$$y(x) = C_1 e^{-\frac{1}{2}x} + C_2 e^{\frac{3}{5}x}$$

where $C_1, C_2 \in \mathbb{R}$ are arbitrary constant.

$$y'(x) = -\frac{1}{2} C_1 e^{-\frac{1}{2}x} + \frac{3}{5} C_2 e^{\frac{3}{5}x}$$

Imposing the ICs: $y(0) = 1$ $\quad y'(0) = 0$

$$1 = y(0) = C_1 + C_2$$

$$0 = y'(0) = -\frac{1}{2} C_1 + \frac{3}{5} C_2$$

$$\begin{cases} C_1 + C_2 = 1 \\ -\frac{1}{2} C_1 + \frac{3}{5} C_2 = 0 \end{cases} \qquad A \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{where } A = \begin{pmatrix} 1 & 1 \\ -\frac{1}{2} & \frac{3}{5} \end{pmatrix}$$

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = A^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\frac{3}{5} + \frac{1}{2}} \begin{pmatrix} \frac{3}{5} & -\cancel{1/4} \\ +\frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} =$$

$$= \frac{1}{\frac{11}{10}} \begin{pmatrix} 3/5 & -1 \\ \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} =$$

$$= \frac{10}{11} \begin{pmatrix} 3/5 & -1 \\ 1/2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 6/11 \\ 5/11 \end{pmatrix}$$

$C_1 = 6/11 \quad C_2 = 5/11 \implies y(x) = \frac{6}{11} e^{-\frac{1}{2}x} + \frac{5}{11} e^{\frac{3}{5}x}$