Queen Mary University of London

MTH5123 Differential Equations, Autumn 2021

Coursework 2 - Week 3 – Selected Solutions

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I. Practice Problems

A. Determine which of the following differential equations are exact differential equations. If an equation is exact, determine its general solution first in implicit form and then (if possible) in explicit form.

1) $(1-y\sin(x))+\cos(x)y'=0$: Denoting $P(x,y)=1-y\sin x, \quad Q(x,y)=\cos x$ we have $\frac{\partial P}{\partial y}=-\sin x=\frac{\partial Q}{\partial x}$, hence the equation is exact. The general solution is looked for in implicit form $F(x,y)=C$, where $F=\int P(x,y)\,dx=\int(1-y\sin x)\,dx=x+y\cos x+g(y)$, where $g(y)$ is to be determined from the condition $Q=\frac{\partial F}{\partial y}=\cos x+g'(y)$. We therefore conclude that $g'(y)=0$ so that $g(y)=const$. Thus the solution in implicit form is $x+y\cos x=C$, whereas the explicit form is $y=(C-x)/\cos x$.

2) $\frac{x}{\sqrt{x^2+y^2}}+\frac{y}{\sqrt{x^2+y^2}}y'=0$: Denoting $P=x/\sqrt{x^2+y^2}, \quad Q=y/\sqrt{x^2+y^2}$ we have $\frac{\partial P}{\partial y}=-\frac{xy}{(x^2+y^2)^{3/2}}=\frac{\partial Q}{\partial x}$, hence the equation is exact. The general solution can be looked for in implicit form $F(x,y)=C$, where

$$F=\int P(x,y)\,dx=\int\frac{x}{\sqrt{x^2+y^2}}\,dx=\sqrt{x^2+y^2}+g(y),$$

where $g(y)$ is to be determined from the condition $Q=\frac{\partial F}{\partial y}=y/\sqrt{x^2+y^2}+g'(y)$. We therefore conclude that $g'(y)=0$ so that $g(y)=const$. Thus the solution in implicit form is $\sqrt{x^2+y^2}=C>0$, whereas explicitly $y=\pm\sqrt{C^2-x^2}$ for $|x|<C$.

3) $-x+(x-y)y'=0$: Denoting $P=-x$ and $Q=x-y$ we have $\frac{\partial P}{\partial y}=0\neq\frac{\partial Q}{\partial x}=1$, so the equation is not exact.

4) $x^2+y/x+y'\ln|xy|=0$: Denoting $P=x^2+y/x, \quad Q=\ln|xy|$ we have $\frac{\partial P}{\partial y}=\frac{1}{x}=\frac{\partial Q}{\partial x}$, so the equation is exact. The general solution can be looked for in implicit form $F(x,y)=C$, where

$$F=\int P(x,y)\,dx=\int\left(x^2+y/x\right)\,dx=\frac{x^3}{3}+y\ln|x|+g(y),$$

where $g(y)$ is to be determined from the condition $Q=\frac{\partial F}{\partial y}=\ln|x|+g'(y)$. We conclude that $g'(y)=\ln|y|$ so that $g(y)=-y+y\ln|y|+const$. The solution in implicit form is given by $\frac{x^3}{3}+y\ln|x|-y+y\ln|y|=C$. It is however impossible to find $y(x)$ explicitly.

5) $2xy^2+4x^3+2(x^2+1)yy'=0$: Denoting $P=2xy^2+4x^3, \quad Q=2(x^2+1)y$ we have $\frac{\partial P}{\partial y}=4xy=\frac{\partial Q}{\partial x}$ so the equation is exact. The general solution can be looked for in implicit form $F(x,y)=C$, where

$$F=\int P(x,y)\,dx=\int(2xy^2+4x^3)\,dx=x^2y^2+x^4+g(y)\,,$$

where $g(y)$ is to be determined from the condition $Q=\frac{\partial F}{\partial y}=2x^2y+g'(y)$. We conclude that $g'(y)=2y$ so that $g(y)=y^2+const$. The solution in the implicit form is given by $x^2y^2+x^4+y^2=C$ or $y(x)=\pm\sqrt{\frac{C-x^4}{1+x^2}}$ explicitly.

B. Please note that the following solution writeups should serve only as an outline or guide to your written justifications and not as a template or model for your answers.

1) Find a value of the parameter b such that the following differential equation is exact and solve it for that value of the parameter:

$$\frac{y-x\,b}{yx}+\frac{x}{y^2}y'=0\,.$$

Solution: Denoting $P(x,y)=\frac{y-x\,b}{yx}, \quad Q(x,y)=\frac{x}{y^2}$ we have $\frac{\partial P}{\partial y}=\frac{yx-x(y-bx)}{y^2x^2}=\frac{b}{y^2}$, whereas $\frac{\partial Q}{\partial x}=\frac{1}{y^2}$, hence only when $b=1$ the equation is exact. For such value of b the general solution is looked for in implicit form $F(x,y)=C$, where

$$F=\int P(x,y)\,dx=\int\frac{y-x}{yx}\,dx=\int\left(\frac{1}{x}-\frac{1}{y}\right)\,dx=\ln|x|-\frac{x}{y}+g(y)\,,$$

where $g(y)$ is to be determined from the condition $Q=\frac{\partial F}{\partial y}=\frac{x}{y^2}+g'(y)$. We therefore conclude that $g'(y)=0$ so that $g(y)=const$. Thus the solution in implicit form is $\ln|x|-\frac{x}{y}=C$, whereas the explicit form is $y=x/(\ln|x|-C)$.

2) Find all functions $f(y)$ such that the following differential equation becomes exact:

$$x^2+\frac{f(y)}{xy}+\ln|xy|\,\frac{dy}{dx}=0$$

and solve it in implicit form for a particular choice such that $f(1)=1$.

Solution: Denoting $P(x,y)=x^2+\frac{f(y)}{xy}, \quad Q(x,y)=\ln|xy|$ we have $\frac{\partial P}{\partial y}=\frac{1}{x}\frac{d}{dy}\left(\frac{f(y)}{y}\right)$, whereas $\frac{\partial Q}{\partial x}=\frac{1}{x}$, hence the equation is exact only if $\frac{d}{dy}\left(\frac{f(y)}{y}\right)=1$ or equivalently $f(y)/y=y+C$ which finally gives $f(y)=y^2+Cy$, with any constant C. The condition $f(1)=1+C=1$ makes us to choose $C=0$. For such value of C the general solution is looked for in implicit form $F(x,y)=C$, where

$$F=\int P(x,y)\,dx=\int\left(x^2+\frac{y}{x}\right)\,dx=\frac{x^3}{3}+y\ln|x|+g(y)\,,$$

where $g(y)$ is to be determined from the condition $Q = \frac{\partial F}{\partial y} = \ln|x| + g'(y)$. We therefore conclude that $g'(y) = \ln|y|$ so that $g(y) = \int \ln|y|\, dy = y\ln|y| - y$. Thus the solution in implicit form is $\frac{x^3}{3} + y\ln|x| + y\ln|y| - y = C$.

C. Determine the general solution of the exact differential equation

$$1 - \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2}y' = 0 \,.$$

Write down the explicit solution for the initial condition $y(0) = e$.

Solution: Denoting

$$P(x,y) = 1 - \frac{x}{x^2+y^2}, \quad Q(x,y) = -\frac{y}{x^2+y^2}$$

we have

$$\frac{\partial P}{\partial y} = \frac{2yx}{(x^2+y^2)^2} = \frac{\partial Q}{\partial x} \,,$$

hence the equation is exact.

The general solution is looked for in implicit form $F(x,y) = C$, where

$$F = \int P(x,y)\, dx = \int \left(1 - \frac{x}{x^2+y^2}\right) dx = x - \frac{1}{2}\ln\left(x^2+y^2\right) + g(y) \,,$$

where $g(y)$ is to be determined from the condition

$$Q = \frac{\partial F}{\partial y} = -\frac{y}{x^2+y^2} + g'(y) \,.$$

We therefore conclude that $g'(y) = 0$ so that $g(y) = const$. Thus the solution in implicit form is

$$x - \frac{1}{2}\ln\left(x^2+y^2\right) = C \,.$$

This can be further rewritten as

$$e^{2(x-C)} = x^2 + y^2 \,,$$

which gives the explicit solution

$$y = \pm\sqrt{e^{2(x-C)} - x^2} \,.$$

We have $y(0) = \pm e^{-C} = e$, hence we choose the plus sign and $C = -1$ and finally obtain

$$y = \sqrt{e^{2(x+1)} - x^2} \,.$$

D. Consider the initial value problem

$$\frac{dy}{dx} = f(x,y), \quad f(x,y) = \sqrt{y^2+p^2}, \quad y(1) = 0 \,,$$

where $p > 0$ is a real parameter. Show that the Picard-Lindelöf Theorem ensures the uniqueness and existence of a solution to the above problem in a rectangular domain $|x-1| \leq A$, $|y| \leq B$. Find the value of the Lipschitz constant K for the above problem for a given A and B. Write down the maximal value of the width A for a given value of B.

Solution: The right-hand side $f(x,y)$ is continuous everywhere, and its derivative $\frac{\partial f}{\partial y}$ satisfies

$$\left|\frac{\partial f}{\partial y}\right| = |y|/\sqrt{p^2+y^2} < 1\,,$$

so is bounded. In our case of initial conditions $a = 1$ and $b = y(1) = 0$, hence in the rectangular domain

$$\mathcal{D} = (|x-1| \leq A,\, |y| \leq B)$$

the solution to the ODE exists and is unique provided $A < B/M$ with $M = \max_{\mathcal{D}} \sqrt{y^2+p^2}$. The value of the Lipschitz constant in such a domain is

$$K = \max_{\mathcal{D}} \left|\frac{\partial f}{\partial y}\right| = \max_{\mathcal{D}} \left|\frac{y}{\sqrt{y^2+p^2}}\right|\,,$$

that is, we should look for a maximum in the interval $-B < y < B$. The function to be maximized is even and as we have

$$\frac{d}{dy}\left(\frac{y}{\sqrt{y^2+p^2}}\right) = \frac{\sqrt{y^2+p^2} - \frac{y^2}{\sqrt{y^2+p^2}}}{y^2+p^2} = \frac{p^2}{(p^2+y^2)^{3/2}} > 0$$

it is growing with y. Hence the maximum is achieved at the end of the interval for $y = \pm B$, and the Lipschitz constant in such a domain is given by

$$K = \frac{B}{\sqrt{B^2+p^2}}\,.$$

Finally, as for a given B we have

$$M = \sqrt{p^2+B^2}$$

(as the function $f(x,y) = \sqrt{p^2+y^2}$ obviously grows with y) this implies that the width A should satisfy

$$A < \frac{B}{M} = \frac{B}{\sqrt{p^2+B^2}}\,.$$

Therefore the maximal value of the width is

$$A = \frac{B}{\sqrt{p^2+B^2}}\,.$$