System of 1^{st}-order ODEs

Suppose to have n-dependent variables

$y_1(x), y_2(x), y_3(x) \dots \qquad y_n(x)$

each dependent variable is a function of a SINGLE independent variable x

A system of 1^{st}-order ODEs is a set of n 1^{st}-order differential equations (ODE) whose normal form reads.

$$
\begin{aligned}
y_1'(x) &= f_1(x, y_1, y_2, \dots y_n) \\
y_2'(x) &= f_2(x, y_1, y_2, \dots y_n) \\
&\vdots \\
y_n'(x) &= f_n(x, y_1, y_2, \dots y_n)
\end{aligned}
$$

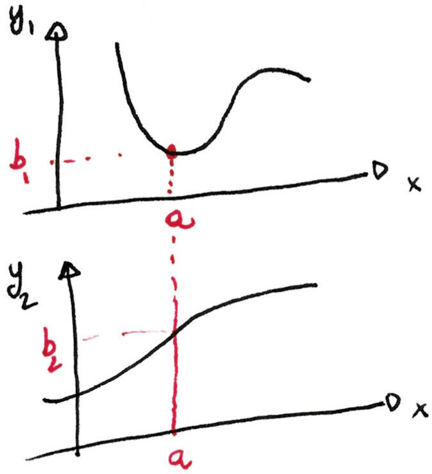
In matrix form $(n=2$

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} f_1(x, y_1, y_2) \\ f_2(x, y_1, y_2) \end{pmatrix}$$

The I.V.P. for a system of $n=2$ ODEs comprises of

System of ODE:
$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} f_1(x, y_1, y_2) \\ f_2(x, y_1, y_2) \end{pmatrix} \qquad (1)$$

ICs:
$$y_1(a) = b_1 \qquad y_2(a) = b_2 \qquad (2)$$



<u>Generalized Picard-Lindelöf theorem</u>

The IVP given by (1) an (2) has one and only one solution in a cuboid D of the form $|x-a| \le A$ $\quad |y_1 - b_1| \le B_1$, $|y_2 - b_2| \le B_2$

if

a) both functions $f_1(x, y_1, y_2)$ & $f_2(x, y_1, y_2)$ are continuous in D

b) the partial derivatives $\dfrac{\partial f_1}{\partial y_1}$ $\quad \dfrac{\partial f_2}{\partial y_1}$ are continuous in D.

$$\frac{\partial f_1}{\partial y_2} \qquad \frac{\partial f_2}{\partial y_2}$$

The equivalence between a single n-order ODE and a system of n 1^{st}-order ODEs

Consider a single n-order ODE

$$y^{(n)} = F(x, y, y', y'', \dots y^{(n-1)}) \qquad (3)$$

This single n-order ODE is equivalent to a system of n 1^{st}-order ODEs

$$\begin{cases} y_1'(x) = y_2(x) \\ y_2'(x) = y_3(x) \\ y_3'(x) = y_4(x) \\ \vdots \\ y_{n-1}'(x) = y_n(x) \\ y_n'(x) = F(x, y_1, y_2, y_3, \dots y_n) \end{cases}$$

Proof: Let us define the functions

$y_1(x) = y(x)$, $y_2(x) = y'(x)$, $y_3(x) = y''(x)$, ... $y_n(x) = y^{(n-1)}(x)$

or

$$\begin{cases} y_1(x) = y(x) \\ y_2(x) = y'(x) \\ y_3(x) = y''(x) \\ \vdots \\ y_{n-1} = y^{(n-2)}(x) \\ y_n = y^{(n-1)}(x) \end{cases}$$

Differentiations

we set $\Rightarrow$

$$\begin{cases} y_1'(x) = y'(x) = y_2 \\ y_2'(x) = y''(x) = y_3 \\ \vdots \\ y_{n-1}' = y^{(n-1)}(x) = y_n \\ y_n' = y^{(n)}(x) = F(x, y_1, y_2 \cdots y_n) \end{cases}$$

$\square$

For a second order ODE we can use the methods that

(A) we will discuss in weeks 4-6

(methods for solving 2^{nd}-order ODEs)

(B) we will discuss in weeks 8-12

(methods for solving a system of $n=2$ 1^{st}-order ODEs)

Example Reduce the following 2^{nd}-order ODE to a system of $n=2$ 1^{st}-order ODEs

$$y'' = 6y - 4y' \qquad y'' = F(x, y', y'') = 6y - 4y'$$

We define $y_1(x)$ and $y_2(x)$

$$\begin{cases} y_1(x) = y(x) \\ y_2(x) = y'(x) \end{cases}$$

$$\begin{cases} y_1'(x) = y'(x) = y_2 \\ y_2'(x) = y''(x) = 6y - 4y' = 6y_1 - 4y_2 \end{cases}$$

System of 1^{st}-order ODE equivalent to the given 2^{nd}-order ODE

$$\begin{cases} y_1'(x) = y_2 \\ y_2''(x) = 6y_1 - 4y_2 \end{cases}$$

The obtained system of n 1^{st}-order ODEs is expressed in terms of the variable x but it can be also transformed into a dynamical system, i.e. a system of n 1^{st}-order ODEs with independent variable t upon identifying $x \equiv t$.

With this transformation

$$y_1(x) = y_1(t) \qquad y_2(x) = y_2(t) \quad \cdots \quad y_n(x) = y_n(t)$$

Additional we have

$$\frac{dy_1(x)}{dx} \to \frac{dy_1(t)}{dt} = \dot{y}_1$$

$$\frac{dy_2(x)}{dx} \to \frac{dy_2(t)}{dt} = \dot{y}_2$$

$$\vdots$$

$$\frac{dy_n(x)}{dx} \to \frac{dy_n(t)}{dt} = \dot{y}_n = F(t, y_1, y_2, \cdots y_n)$$

Therefore the n-order ODE

$$y^{(n)} = F(x, y, y', y'' \dots y^{(n-1)})$$

is equivalent to the system $(x \equiv t)$

$$\begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_2 = y_3 \\ \vdots \\ \dot{y}_{n-1} = y_n \\ \dot{y}_n = F(t, y_1, y_2, y_3 \dots y_n) \end{cases}$$

Example $\quad y'' = 6y - 4y'$

Previously we have shown the equivalence to the system of $n = 2$ 1^{st}-order ~~ODE~~ ODEs

$$\begin{cases} y_1' = y_2 \quad (x) \\ y_2' = 6y_1 - 4y_2 \end{cases}$$

By putting $x \equiv t$ we derive the equivalence to

$$\begin{cases} \dot{y}_1(t) = y_2(t) \\ \dot{y}_2(t) = 6y_1 - 4y_2 \end{cases}$$

$\square$