

Picard-Lindelöf theorem.

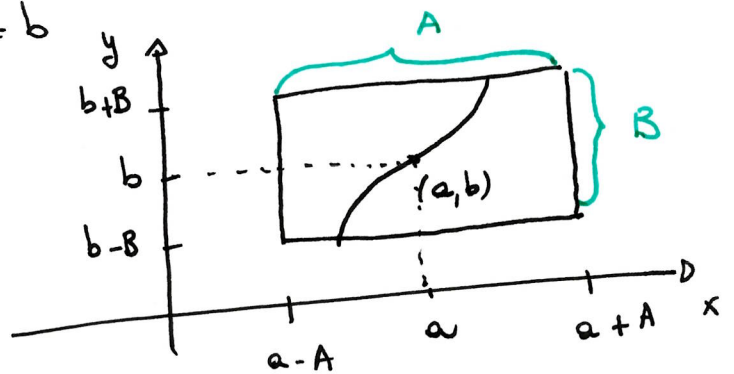
Introduction

An initial value problem (IVP) for a 1st-order ODE

comprises

$$\text{ODE: } y' = f(x, y)$$

$$\text{I.C.: } y(a) = b$$



The Picard-Lindelöf theorem provides the SUFFICIENT conditions for the existence and uniqueness of the solution to the IVP in a rectangular region D:

$$|x - a| \leq A$$

$$|y - b| \leq B$$

for $A > 0$, $B > 0$

Picard - Lindelöf theorem

Consider the IVP

$$y' = f(x, y) \quad \& \quad y(a) = b$$

This IVP has one and only one solution in a rectangular region D : $|x - a| \leq A$, $\& \ |y - b| \leq B$ with $A > 0$, $B > 0$ provided the following conditions ~~are~~ hold:

- The function $f(x, y)$ is continuous in D and therefore is bounded in D with
 $|f(x, y)| \leq M \quad \forall (x, y) \in D \quad \text{with } M > 0$

M is related to A and B as we must have

$$A \leq \frac{B}{M}$$

- Lipshitz condition

The partial derivative $\frac{\partial}{\partial y} f(x, y)$ is bounded in D

that is $K = \max_{(x, y) \in D} \left| \frac{\partial f}{\partial y} \right|$ with $0 < K < \infty$

K is called the Lipshitz constant.

(ensures uniqueness)

Note: if $\frac{\partial}{\partial y} f(x, y)$ is continuous in D , then the Lipshitz condition is met, because it is necessarily bounded.

Example

$$y' = \frac{1}{2y} \quad \& \quad y(0) = 0$$

ODE: $y' = f(x, y) = \frac{1}{2y}$

$$f(x, y) = \frac{1}{2y}$$

I.C. $y(a) = b$

$$a = 0, \quad b = 0$$

Rectangular region D:

$$|x - a| \leq A$$

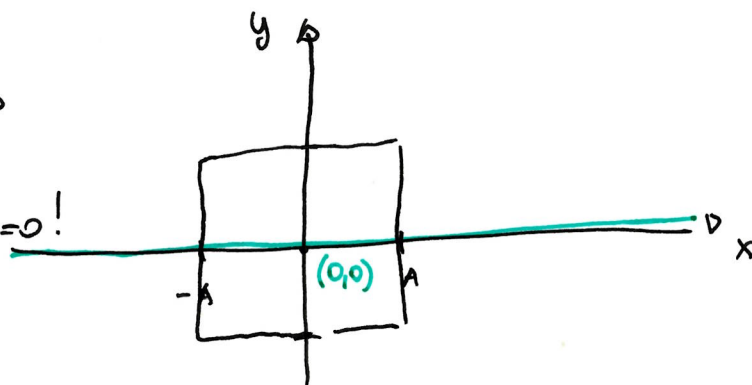
$$|y - b| \leq B$$

$$\Rightarrow \begin{cases} |x| \leq A \\ |y| \leq B \end{cases}$$

with
 $A > 0$
 $B > 0$

- $f(x, y)$ should be continuous in D

$f(x, y) = \frac{1}{2y}$ is NOT continuous in $y=0$!



The hypothesis & hypothesis of the Picard-Lindelöf theorem do NOT hold.

Example

$$y' = 3y^{2/3}$$

$$\& \quad y(0) = 0$$

ODE: $y' = f(x, y) = 3y^{2/3}$

$$f(x, y) = 3y^{2/3}$$

I.C. $y(a) = b$

$$a = 0, \quad b = 0$$

Rectangular region D:

$$|x - a| \leq A$$

$$|y - b| \leq B$$

$$\Rightarrow \begin{cases} |x| \leq A \\ |y| \leq B \end{cases}$$

with $A > 0$
 $B > 0$

- $f(x, y)$ is continuous in D

$$f(x, y) = 3y^{2/3}$$

continuous in $|x| \leq A, |y| \leq B$ ✓

- $\frac{\partial}{\partial y} f(x, y) = 3 \cdot \frac{2}{3} y^{-1/3} = 2y^{-1/3} = \frac{2}{y^{1/3}}$ diverges for $y \rightarrow 0$

$\frac{\partial f}{\partial y}$ is NOT bounded in D.

The theorem cannot ensure uniqueness of the solution!

Example $y' = x^2 |y|^{1/3}$ & $y(0) = 1$

ODE: $y' = f(x, y) = x^2 |y|^{1/3}$ $f(x, y) = x^2 |y|^{1/3}$

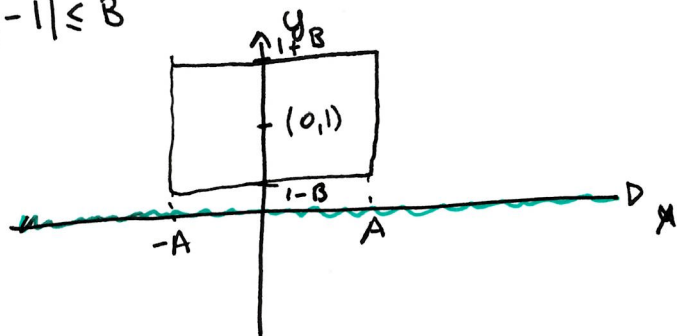
I.C.: $y(a) = b$ $a = 0, b = 1$

Rectangular region D

$$\begin{aligned} |x-a| &\leq A \\ |y-b| &\leq B \end{aligned} \quad \text{with } A > 0, B > 0$$

$$\begin{aligned} |x| &\leq A \\ |y-1| &\leq B \end{aligned}$$

with $AB > 0$



① Let us check whether $f(x, y)$ is continuous in D.

$$f(x, y) = \begin{cases} x^2 y^{1/3} & \text{if } y > 0 \\ -x^2 y^{1/3} & \text{if } y \leq 0 \end{cases}$$

For checking continuity in $y = 0$

$$\lim_{y \rightarrow 0^+} x^2 y^{1/3} = \lim_{y \rightarrow 0^-} -x^2 y^{1/3} = 0$$

$f(x, y)$ is continuous in D ✓

② Lipschitz condition

$$\frac{\partial f}{\partial y} = \begin{cases} x^2 \frac{1}{3} y^{-2/3} & \text{if } y > 0 \\ -x^2 \frac{1}{3} y^{-2/3} & \text{if } y \leq 0 \end{cases} \quad \text{diverges for } y \rightarrow 0$$

The rectangular region

$$\begin{aligned} |x| \leq A & \Rightarrow -A \leq x \leq A \\ |y-1| \leq B & \Rightarrow 1-B \leq y \leq 1+B \end{aligned}$$

We must impose

$$y > 0$$

$$0 < 1-B \leq y \leq 1+B$$

$$\boxed{0 < B < 1}$$

Under the condition $0 < B < 1$ $\frac{\partial f}{\partial y}$ is bounded in D ✓

③ We need to impose $A \leq \frac{B}{M}$ where $M = \max_{(x,y) \in D} |f(x,y)|$

$$\text{We know that } M = \max_{(x,y) \in D} |f(x,y)| = A^2 (1+B)^{1/3}$$

$$M = \max_{(x,y) \in D} |x^2 y^{1/3}| = \max_{-A \leq x \leq A} x^2 \cdot \max_{1-B \leq y \leq 1+B} |y|^{1/3}$$

Calculating the max will not be examinable.

$$\text{We need to impose } A \leq \frac{B}{M} = \frac{B}{A^2 (1+B)^{1/3}}, \quad A \leq \frac{B}{A^2 (1+B)^{1/3}}$$

$$\Rightarrow A^3 \leq \frac{B}{(1+B)^{1/3}} \Rightarrow 0 < A \leq \frac{B^{1/3}}{(1+B)^{1/9}}$$

In the region

$$|x| \leq A$$

$$|y-r| \leq B$$

with

$$0 < A \leq \frac{B^{1/3}}{1+B^{1/9}}$$

$$0 < B < 1$$

the solution to the IVP exist and is unique .