Motivation for the Picard - Lindelöf theorem
(existence and uniqueness of solution to IVP)

Initial value problem I.V.P. for 1^{st}-order ODEs comprises of

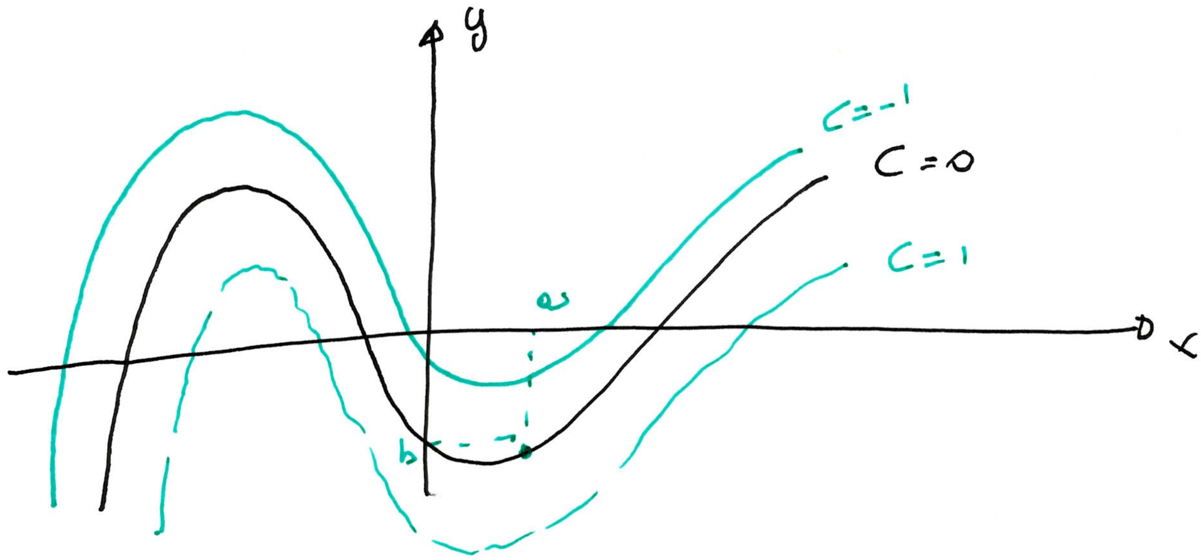
1) ODE: $y' = f(x,y)$

2) I.C. $y(a) = b$

The ODE has a general solution that depends on an arbitrary constant C

$$\boxed{F(x, y(x)) = C}$$ Implicit general solution

The I.C. imposes that the solution to the IVP. pass through the point (a,b). You look for solutions $y = y(x)$ to the ODE $y' = f(x,y)$ that satisfy $y(a) = b$.

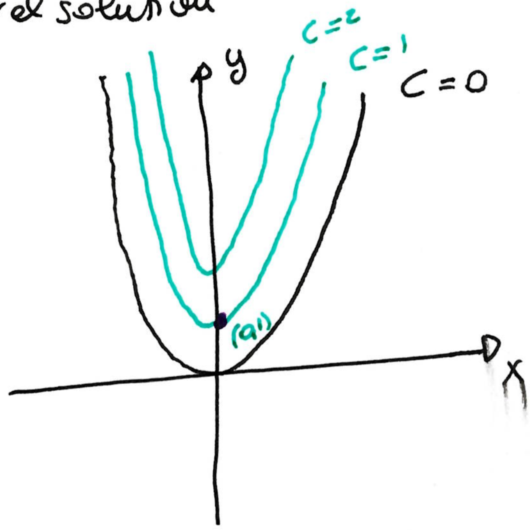


Example

$$y' = x \quad \& \quad y(0) = 1$$

$$y(x) = \int x \, dx + C = \frac{1}{2} x^2 + C \quad \text{general solution}$$

I.C. $\quad y(0) = 1 \quad$ pass through $(0,1)$



$$1 = y(0) = C \Rightarrow C = 1$$

$$\boxed{y(x) = \frac{1}{2} x^2 + 1}$$

solution to I.V.P.

Example

$$y' = -\frac{y}{x+1} \quad \& \quad y(0) = \cancel{1} - 1$$

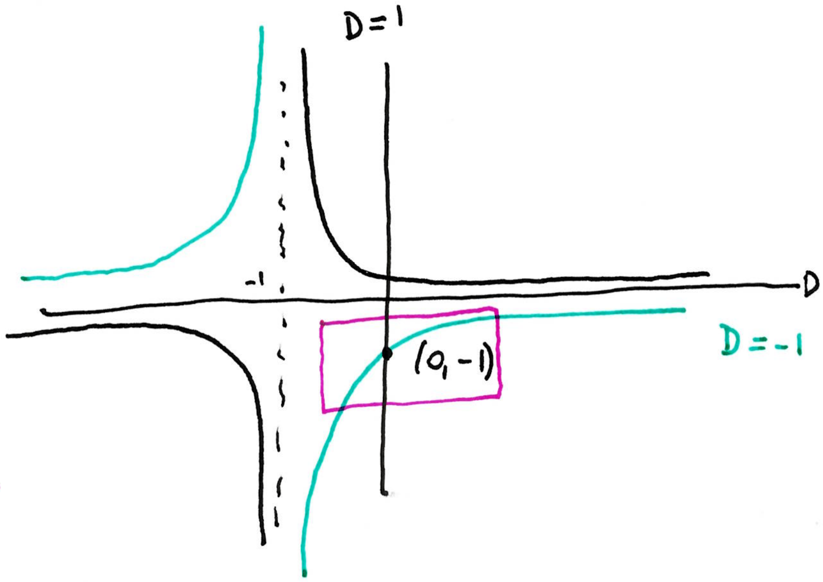
separable

General solution $\quad y(x) = \dfrac{D}{x+1} \quad$ for $x \neq -1$

[check at home]

Impose I.C.

$$-1 = y(0) = D \Rightarrow D = -1$$



Solution

$$\boxed{y(x) = -\frac{1}{x+1}}$$

Locally unique

But is this unique globally? No

$$\text{e.g.} \quad y(x) = \begin{cases} -\dfrac{1}{x+1} & \text{for } x > -1 \\ \dfrac{D}{x+1} & \text{for } x < -1 \end{cases}$$

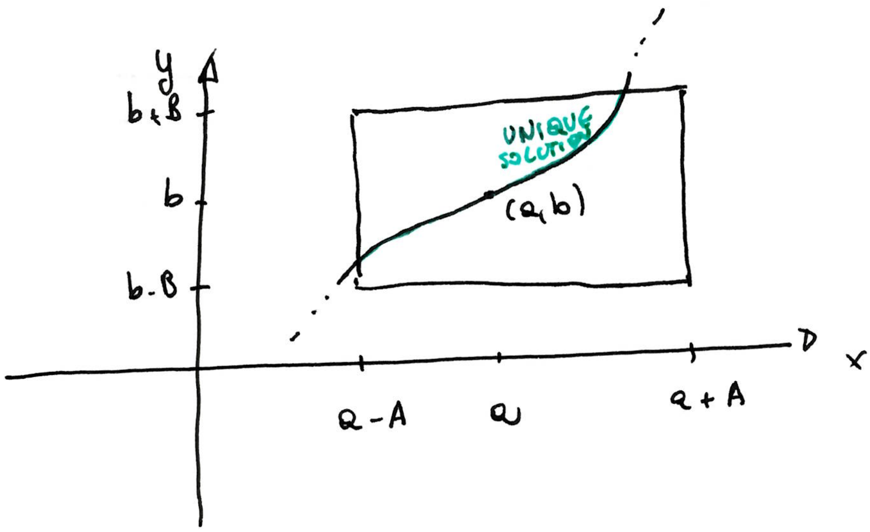
Definition

An initial value problem (IVP) formed by ODE + IC $y(a)=b$ has a

UNIQUE SOLUTION if

for any two solutions $y_1(x)$, $y_2(x)$ satisfying the IVP there exist $A>0$, $B>0$ such that

$$y_1(x)=y_2(x) \qquad \forall x \in (a-A, a+A) \quad \forall y \in (b-B, b+B)$$



However there are cases in which the solution is NOT UNIQUE in any D $|x-a| \leq A$, $|y-b| \leq B$

Example

$$y' = \frac{1}{2y} \quad \& \quad y(0) = b$$

$\frac{dy}{dx} = \frac{1}{2y}$ separable $\quad \int 2y\, dy = \int dx + C'$

$y^2 = x + C'$ implicit solution

$y = \pm\sqrt{x + C'}$ explicit solution

Impose I.C. $\quad y(0) = b$

- If $b > 0$ $\quad b = y(0) = \pm\sqrt{0 + C'} \implies C = b^2$

  $y(x) = +\sqrt{x + b^2}$ Unique solution

- If $b < 0$ $\quad b = y(0) = \pm\sqrt{0 + C'} \implies C = b^2$

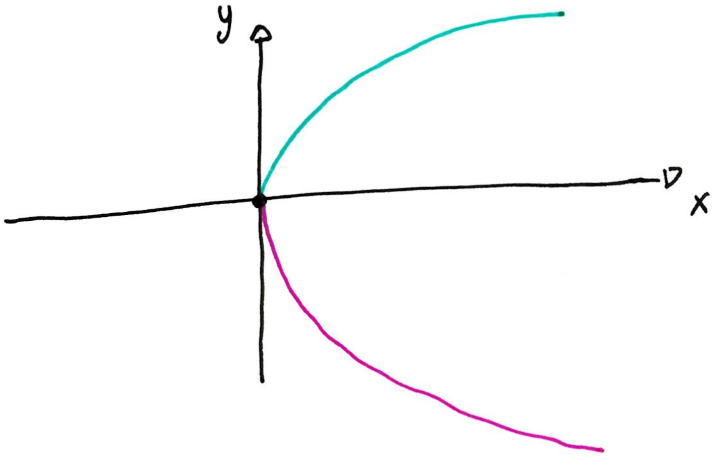
  $y(x) = -\sqrt{x + b^2}$ Unique solution

- If $b = 0$ $\quad 0 = b = y(0) = \pm\sqrt{0 + C'} \implies C = 0$

  Two solutions!

  $\boxed{y(x) = \sqrt{x}}$

  $\boxed{y(x) = -\sqrt{x}}$



Example $\quad y' = 3y^{2/3} \quad \& \quad y(0) = 0$

ODE: $\quad \frac{dy}{dx} = 3y^{2/3} = g(y) f(x) \qquad g(y) = 3y^{2/3} \quad f(x) = 1$

separable

I.C. $\quad y(0) = 0 \ , \ y(a) = b \qquad a = 0, \ b = 0$

(1) Solve the ODE by separation of variables.

$$\int \frac{dy}{3y^{2/3}} = \int dx + C'$$

LHS: $\quad H(y) = \int \frac{dy}{3y^{2/3}} = \int \frac{1}{3} y^{-2/3} dy = y^{1/3}$

RHS: $\quad F(x) = \int dx = x$

$H(y) = F(x) + C' \quad \Rightarrow \quad y^{1/3} = x + C'$

$$\boxed{y = (x + C')^3}$$ Explicit solution

(2) Check for constant solutions

$g(y) = 3y^{2/3} = 0 \qquad y = 0 \quad$ root of $g(y)$

$\Rightarrow \boxed{y(x) = 0}$ is a ~~So~~ solution to the ODE

Let us impose the I.C. $y(0)=0$

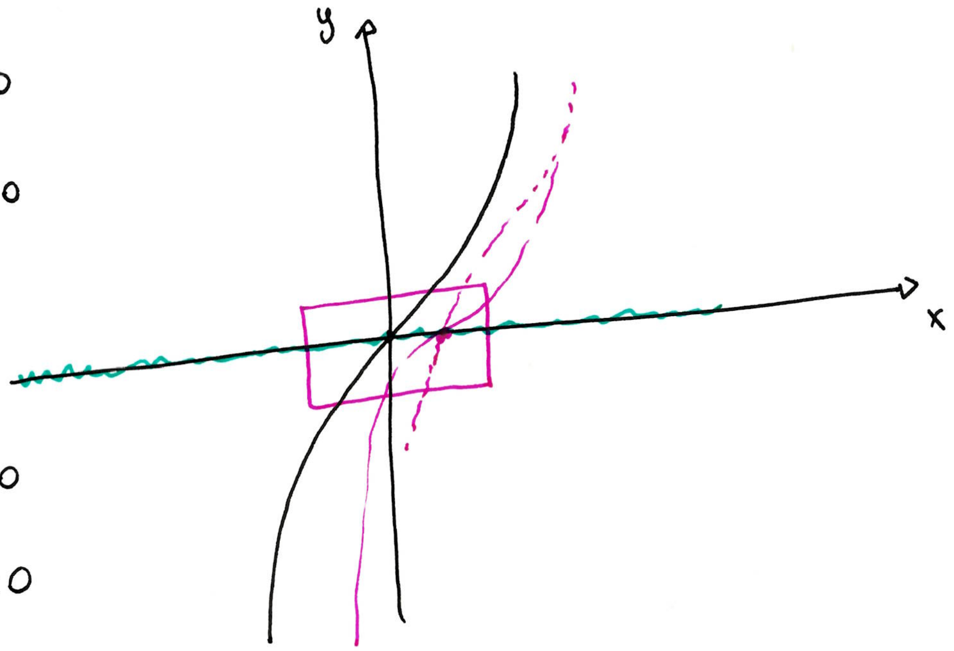
① $0 = y(0) = (0+c)^3 = c^3 \Rightarrow c=0$

$y(x)=x^3$ is a solution to the IVP.

② $0 = y(0)$ satisfied ~~fo~~ by $y(x)=0$

$y(x)=0$ is a solution to the I.V.P.

$$y(x) = \begin{cases} 0 & \text{if } x<0 \\ x^3 & \text{if } x \geq 0 \end{cases}$$



$$y(x) = \begin{cases} 0 & \text{if } x \geq 0 \\ x^3 & \text{if } x<0 \end{cases}$$

If I.C. is ~~x=a~~ $y(a)=0$ we have solution

$$\begin{cases} y(x)=0 & \text{is a solution} \\ y(x)=(x-a)^3 & \text{is a solution} \end{cases}$$

The IVP has infinite solutions!