Formative assessment #3

A.1) Determine if the following equation is exact. If ~~it~~ is exact determine the general solution first in implicit form and then (if possible) in explicit form.

$$\begin{cases} (1 - y\sin x) + (\cos x)\,y' = 0 \\ P(x,y) + Q(x,y)\,y' = 0 \end{cases}$$

$P(x,y) = \frac{\partial F}{\partial x}$ $\qquad Q(x,y) = \frac{\partial F}{\partial y}$

$P(x,y) = 1 - y\sin x$

$F(x,y) = C$ Implicit solution.

$Q(x,y) = \cos x$

The ODE is exact if and only if

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}F\right) = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}F\right)$$

$$\frac{\partial}{\partial x}Q(x,y) = \frac{\partial}{\partial y}P(x,y)$$

$$\frac{\partial}{\partial x}Q(x,y) = \frac{\partial}{\partial x}\cos x = -\sin x \checkmark$$

Yes! The ODE is exact.

$$\frac{\partial}{\partial y}P(x,y) = \frac{\partial}{\partial y}(1 - y\sin x) = -\sin x \checkmark$$

Integrate

$$F(x,y) = \int P(x,y)\,dx + g(y) = \int (1 - y\cos x)\,dx + g(y)$$

$$F(x,y) = x - y\int \sin x\,dx + g(y) = x + y\cos x + g(y)$$

$$\frac{\partial F}{\partial y} = Q(x,y) = \cos x$$

LHS: $\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left[x + y\cos x + g(y)\right] = \cos x + g'(y)$

RHS: $\cos x = Q(x,y)$

$$\cos x = \cos x + g'(y) \quad \Rightarrow \quad g'(y) = 0$$

$$g(y) = C_1$$

$$F(x,y) = x + y\cos x + C_1 = C$$

$$\boxed{x + y\cos x = D}$$

where $D = C - C_1$ is an arbitrary constant

Implicit solution

$$\boxed{y = -\frac{(x-D)}{\cos x}}$$

Explicit solution

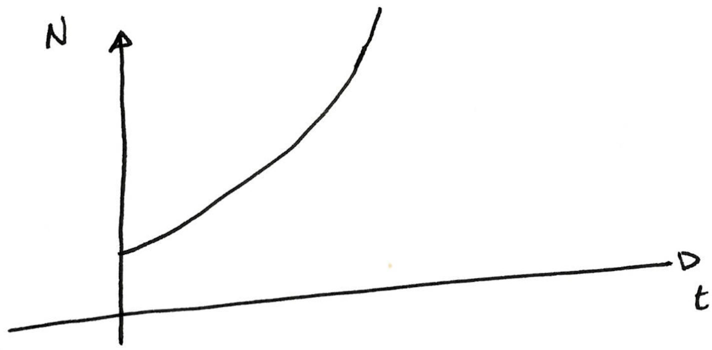
Application of ODEs to ecosystems modelling.

$N(t)$ · (average / expected) number of individuals of a species at time t.

1) $\dot{N} = \lambda N$ where $\lambda > 0$ is the reproduction rate

$N(0) = 1$ show that it leads to exponential growth.

$$N(t) = e^{\lambda t}$$



But in reality often there are limited resources.

In this case one puts

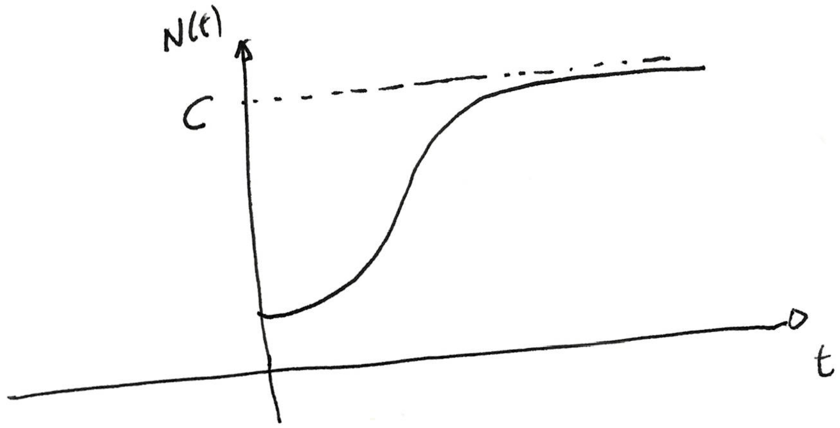
$$\lambda \to \lambda \left(1 - \frac{N}{C}\right)$$

the reproduction rate is reduced if the population is larger (competition for resources)

where $C \in \mathbb{R}$ is a constant called carrying capacity)

$$\dot{N} = \lambda \left(1 - \frac{N}{C}\right) N \qquad N(0) = 1 \quad \& \quad C > 1$$

show that it leads to $N(t) = \dfrac{C e^{\lambda t}}{C - 1 + e^{\lambda t}}$



The population saturates to the carrying capacity.

Solution

$\dot{N} = \lambda N$ separable

$\frac{dN}{dt} = \lambda N \Rightarrow \int \frac{dN}{N} = \int \lambda \, dt + C$

$\ln|N| = \lambda t + C$

$|N| = e^{\lambda t + C} \Rightarrow N = \underbrace{\boxed{\pm e^{C}}}_{D} e^{\lambda t}$

① $N(t) = De^{\lambda t}$ where $D \in \mathbb{R}$ is an arbitrary constant

Is there another constant solution?

② Yes $N(t) = 0$

Impose I.C. $N(0) = 1$ Only ① Needs to be considered

$1 = N(0) = D \Rightarrow \boxed{D = 1}$ Therefore $N(t) = e^{\lambda t}$

$$\dot{N} = \lambda\left(1 - \frac{N}{C}\right) N \qquad \& \quad N(0) = 1$$

separable

$$\frac{dN}{dt} = \lambda\left(1 - \frac{N}{C}\right) N \quad \Rightarrow \int \frac{dN}{N\left(1-\frac{N}{C}\right)} = \int \lambda \, dt + C_1'$$

$$\frac{1}{N\left(1-\frac{N}{C}\right)} = \frac{A}{N} + \frac{B}{1-\frac{N}{C}} = \frac{A\left(1-\frac{N}{C}\right) + BN}{N\left(1-\frac{N}{C}\right)} = \frac{A + N\left(B - A/C\right)}{N\left(1-\frac{N}{C}\right)}$$

$$\left.\begin{array}{l} B - A/C = 0 \\ A = 1 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} A = 1 \\ B = 1/C \end{array}\right.$$

$$H(N) = \int \frac{dN}{N\left(1-\frac{N}{C}\right)} = \int \left[\frac{1}{N} + \frac{1}{C}\frac{1}{\left(1-\frac{N}{C}\right)}\right] dN = \int \frac{1}{N}\, dN + \int \frac{1}{C-N}\, dN$$

$$= \ln|N| - \ln|C-N|$$

$$F(t) = \int \lambda \, dt = \lambda t$$

$$H(N) = F(t) + C_1' \qquad \text{Implicit solution}$$

$$\ln\left|\frac{N}{C-N}\right| = \lambda t + C_1' \qquad \left|\frac{N}{C-N}\right| = e^{\lambda t + C_1'}$$

$$\frac{N}{C-N} = \underbrace{\boxed{\pm e^{C_1'}}}_{= D} e^{\lambda t}$$

$$\frac{N}{C-N} = De^{\lambda t}$$

$$N = De^{\lambda t}(C - N)$$

$$N(1 + De^{\lambda t}) = De^{\lambda t} \cdot C'$$

$$N = \frac{De^{\lambda t} \cdot C'}{1 + De^{\lambda t}}$$

where $C' > 1$ is the carrying capacity

while $D \in \mathbb{R}$ is the arbitrary constant.

Imposing the IC. $N(0) = 1$

$$1 = N(0) = \frac{C' \cdot D}{1 + D} \implies 1 + D = C' D$$

$$\implies D(1 - C') = -1$$

$$D = \frac{-1}{1 - C'}$$

$$N(t) = \frac{C' e^{\lambda t}}{-1 + C' + e^{\lambda t}}$$

$$\lim_{t \to \infty} N(t) = \lim_{t \to \infty} \frac{C' e^{\lambda t}}{-1 + C' + e^{\lambda t}} = C'$$

Asymptotically in time the population will saturate to the carrying capacity.