Exact 1^{st}-order ODE

Exact 1^{st}-order ODEs are ODEs of the type

$$P(x,y) + Q(x,y)\frac{dy}{dx} = 0 \qquad (1)$$

which admits an implicit solution

$$F(x, y(x)) = C \qquad (2)$$

where $C \in \mathbb{R}$ is an arbitrary constant.

We apply the chain rule to Eq (2) we get

$$\frac{d}{dx}F(x,y(x)) = \underbrace{\boxed{\frac{\partial}{\partial x}F(x,y)}}_{= P(x,y)} + \underbrace{\boxed{\frac{\partial F(x,y)}{\partial y}}}_{= Q(x,y)}\frac{dy}{dx} = 0$$

which reduces to Eq. (1) if and only if

$$\boxed{P(x,y) = \frac{\partial}{\partial x}F(x,y) \quad \& \quad Q(x,y) = \frac{\partial}{\partial y}F(x,y)}$$

If F is twice differentiable.

$$\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x}$$

$$\frac{\partial}{\partial x}\frac{\partial F}{\partial y} = \frac{\partial}{\partial x} Q(x,y) = \frac{\partial}{\partial y} \underbrace{\boxed{\frac{\partial F}{\partial x}}}_{= P(x,y)} = \frac{\partial}{\partial y} P(x,y)$$

$$\boxed{\frac{\partial}{\partial x} Q(x,y) = \frac{\partial}{\partial y} P(x,y)} \qquad (3)$$

must hold

> It follows that Eq. (3) is the condition ensuring that Eq. (1) is exact

Example Check whether the following ODE is exact

$$(3x^2+y) = (3y^2-x)\frac{dy}{dx} \qquad (*)$$

(1) We put Eq $*$ in the form $P(x,y) + Q(x,y)\frac{dy}{dx} = 0$

$$\underbrace{(3x^2+y)}_{P(x,y)} \underbrace{- (3y^2-x)}_{Q(x,y)} \frac{dy}{dx} = 0$$

$P(x,y) = 3x^2 + y$

$Q(x,y) = -3y^2 + x$

$$\frac{\partial}{\partial x} Q(x,y) = \frac{\partial}{\partial y} P(x,y)$$

$1 = \frac{\partial}{\partial y} P(x,y) = \frac{\partial}{\partial x} Q(x,y)$

LHS: $\frac{\partial}{\partial x} Q(x,y) = \frac{\partial}{\partial x}(-3y^2+x) = 1$

RHS: $\frac{\partial}{\partial y} P(x,y) = \frac{\partial}{\partial y}(3x^2+y) = 1$

The ODE is EXACT!!

Solution of an exact ODE.

An exact 1^{st}-order ODE is an ODE of the type

$$P(x,y) + Q(x,y)\frac{dy}{dx} = 0 \qquad (1)$$

~~which~~ with implicit solution $F(x,y) = C$ where

$$\boxed{P(x,y) = \frac{\partial}{\partial x} F(x,y)} \quad \& \quad \boxed{Q(x,y) = \frac{\partial}{\partial y} F(x,y)}$$

We need to find the expression of F

In order to find $F(x,y)$ we integrate

$$P(x,y) = \frac{\partial}{\partial x} F(x,y)$$

finding

$$\boxed{F(x,y) = \int P(x,y)\,dx + g(y) \qquad (4)}$$

where $g(y)$ is an arbitrary function of y.

To determine $g(y)$ we use.

$$Q(x,y) = \frac{\partial}{\partial y} F(x,y) = \frac{\partial}{\partial y}\left[\int P(x,y)\,dx + g(y)\right]$$

$$Q(x,y) = \frac{\partial}{\partial y}\int P(x,y)\,dx + g'(y)$$

It follows that $g(y)$ satisfies the ODE

$$\boxed{g'(y) = Q(x,y) - \frac{\partial}{\partial y}\int P(x,y)\,dx} \qquad (5)$$

Equation for $g(y)$

We can obtain $g(y)$ by integration of (5)

The solution to Eq (1) is given by

$$\boxed{F(x,y) = \int P(x,y)\,dx + g(y) + C'}$$

where $g(y)$ is the solution to Eq. (5)

Example Solve

$$\underbrace{(3x^2+y)}_{P(x,y)} - \underbrace{(3y^2-x)}_{Q(x,y)}\frac{dy}{dx} = 0$$

1. Identify $P(x,y) = 3x^2+y$

   $Q(x,y) = -3y^2+x$

2. $\frac{\partial}{\partial x}F(x,y) = P(x,y)$

$$F(x,y) = \int P(x,y)\,dx + g(y)$$

$$F(x,y) = \int (3x^2+y)\,dx + g(y) = x^3 + yx + g(y)$$

(3) $$\frac{\partial}{\partial y} F(x,y) = Q(x,y)$$

$$Q(x,y) = \frac{\partial}{\partial y} F(x,y) = \frac{\partial}{\partial y}\left[x^3 + yx + g(y)\right] = x + g'(y)$$

$$Q(x,y) = -3y^2 + x$$

$$-3y^2 + \cancel{x} = \cancel{x} + g'(y) \qquad g'(y) = -3y^2$$

$$g(y) = \int -3y^2\,dy + C_1 = -y^3 + C_1$$

(4) $F(x,y) = C$ implicit solution

$$F(x,y) = x^3 + yx - y^3 + C_1 = C$$

$$x^3 + yx - y^3 = \underbrace{C - C_1} = D$$

where $D \in \mathbb{R}$ is an arbitrary constant

$$\boxed{x^3 + yx - y^3 = D}$$ Implicit solution

Summary of week 1 & 2

We discussed are 6 types of 1st-order ODEs.

1. $y' = f(x)$ calculus

2. $y' = f(x)\,g(y)$ separable.

Reducible to separable

$$\begin{cases} y' = f(ax+by+c) & \text{(3)} \\ y' = f(y/x) \quad \text{scale-invariant} & \text{(4)} \end{cases}$$

5. Linear ODEs

$$y' = A(x)\,y + B(x)$$

a) $B(x) = 0$ homogeneous

$$y' = A(x)\,y \qquad \text{separable}$$

b) $B(x) \neq 0$ inhomogeneous

Variation of parameter method.

6. Exact ODEs (can be non-linear)

$$P(x,y) + Q(x,y)\frac{dy}{dx} = 0$$

Exact if

$$\frac{\partial}{\partial y}P(x,y) = \frac{\partial}{\partial x}Q(x,y)$$