Linear ODEs

A linear ODE of order n is an ODE of the type

$$\boxed{a_n(x)\,y^{(n)}(x) + a_{n-1}(x)\,y^{(n-1)}(x) + \dots + a_2(x)\,y^{(2)}(x) + a_1(x)\,y^{(1)}(x) + a_0(x)\,y_0(x) = f(x)}$$

with $a_n(x) \neq 0$

1) The coefficients $a_i(x)$ are only functions of the INDEPENDENT VARIABLE

2) These ODEs only contain linear combination of the DEPENDENT VARIABLE y and its derivatives $y^{(n)}, y^{(n-1)} \dots y'', y'$

- $y(x)$ and its derivative occur
  - only to the first power
  - they are not multiplied by each other
  - they do not appear in non-linear functions

  $\sin y$, e^y, $\cosh y'$, etc..

Example $$\underbrace{(\ln x)^2}_{a_2(x)} y'' + \underbrace{e^x}_{a_1(x)} y' = \underbrace{\tanh x}_{f(x)}$$

$a_2(x)\,y'' + a_1(x)\,y' + a_0(x)\,y = f(x)$, $a_0 = 0$

2nd-order linear ODE.

$x y''' + e^y = 0$ 3rd-order NON-LINEAR!

$y y' + x = 0$ 1st-order NON LINEAR.

1^{st}-order ~~ode~~ linear ODEs

A first-order linear ODE is a ODE of the type

$$\boxed{a_1(x)\, y' + a_0(x)\, y = f(x)} \qquad \text{where } a_1(x) \neq 0$$

We can express it in normal form as

$$\boxed{y' = A(x)\, y + B(x)} \qquad (1)$$

where $A(x) = -\dfrac{a_0(x)}{a_1(x)}$ and $B(x) = \dfrac{f(x)}{a_1(x)}$

In fact

$$a_1(x)\, y' + a_0(x)\, y = f(x)$$

$$a_1(x)\, y' = -a_0(x)\, y + f(x)$$

$$y' = \underbrace{\boxed{-\frac{a_0(x)}{a_1(x)}}}_{= A(x)}\, y + \underbrace{\boxed{\frac{f(x)}{a_1(x)}}}_{= B(x)}$$

The first-order linear ODE is called

- homogeneous if $B(x) = 0$
- inhomogeneous if $B(x) \neq 0$

1^{st}-order linear ODEs

A first order linear ODE is a ODE of the type

$$\boxed{a_1(x)\, y' + a_0(x)\, y = f(x)}$$

where $a_1(x) \neq 0$

We can express it in normal form as

$$\boxed{y' = A(x)\, y + B(x)} \qquad (1)$$

where $A(x) = -\dfrac{a_0(x)}{a_1(x)}$ $\qquad B(x) = \dfrac{f(x)}{a_1(x)}$

In fact:

$$a_1(x)y' + a_0(x)y = f(x)$$

$$a_1(x)y' = -a_0(x)y + f(x)$$

$$y' = \underbrace{\boxed{-\frac{a_0(x)}{a_1(x)}}}_{=A(x)} y + \underbrace{\boxed{\frac{f(x)}{a_1(x)}}}_{=B(x)} \implies \text{Eq. (1)}$$

$\square$

The first order linear ODEs are called

- homogeneous if $B(x) = 0$
- inhomogeneous if $B(x) \neq 0$

Examples

$y' = A(x)y + B(x)$

$y' = \underbrace{(\sin x)}_{A(x)} y$ 1st-order linear homogeneous ODE.

$B(x) = 0$

$y' = \underbrace{e^x}_{=A(x)} y + \underbrace{x}_{=B(x)}$ 1st-order linear inhomogeneous ODE

$y' = 1 - y^2 + x$ 1st-orde NON-LINEAR ODE.

Solution of 1^{st}-order linear ODE

Step 1 Solving the 1^{st}-order homogeneous equ.

$$y' = A(x)\, y \qquad \text{separable ODE.}$$

$$\frac{dy}{dx} = A(x)\, y \implies \int \frac{dy}{y} = \int A(x)\, dx + C_1$$

$$\implies \ln|y| = \int A(x)\, dx + C_1$$

$$|y| = e^{\int A(x)\, dx + C_1} = e^{C_1} e^{\int A(x) dx}$$

$$y = \underbrace{\boxed{\pm e^{C_1}}}_{=D} e^{\int A(x)\, dx}$$

$$\boxed{y(x) = D e^{\int A(x)\, dx}}$$

General solution of the homogeneous ODE.

where D is an arbitrary constant $D \in \mathbb{R}$.

Step 2 Variation of parameter method

We look for solutions of the inhomogeneous ODE

$$\boxed{y' = A(x)\, y + B(x)} \qquad (1)$$

in a form

$$\boxed{y(x) = D(x)\, e^{\int A(x)\, dx}} \qquad (2)$$

We consider the LHS of Eq. (1) and we plug $y(x)$ given by Eq. (2) in Eq. (1)

$$\text{LHS: } y' = D'(x)\, e^{\int A(x)dx} + D(x) \underbrace{A(x)\, e^{\int A(x)dx}}_{\frac{d}{dx} e^{\int A(x)dx}}$$

We consider the RHS of Eq. (1)

$$\text{RHS: } A(x)\, y(x) + B(x) = A(x) \underbrace{D(x)\, e^{\int A(x)dx}}_{= y(x)} + B(x)$$

Therefore Eq. (1) implies

$$D'(x)\, e^{\int A(x)dx} + \cancel{\underbrace{D(x) A(x)\, e^{\int A(x)dx}}} = \cancel{\underbrace{A(x) D(x)\, e^{\int A(x)dx}}} + B(x)$$

$$D'(x)\, e^{\int A(x)dx} = B(x)$$

$$D'(x) = B(x)\, e^{-\int A(x)dx}$$

$$\boxed{D(x) = \int B(x)\, e^{-\int A(x)dx}\, dx + C}$$

The general solution of the 1-st-order linear inhomogeneous ODE

$$y(x) = D(x)\, e^{\int A(x)\,dx}$$

Inserting the expression for D(x) we obtain

$$\boxed{y(x) = e^{\int A(x)\,dx} \left(\int B(x)\, e^{-\int A(x)\,dx} + C \right)}$$

General solution to the inhomogeneous linear ODE.

We can write $y(x) = y_h(x) + y_p(x)$

where $y_h(x)$ is the general solution to the homogeneous ODE

and $y_p(x)$ is a particular solution to the inhomogeneous ODE.

$$y(x) = \underbrace{C\, e^{\int A(x)\,dx}}_{y_h(x)} + \underbrace{e^{\int A(x)\,dx} \int B(x)\, e^{-\int A(x)\,dx}}_{y_p(x)}$$

Example

$$y' + 2xy = x$$

$$y' = \underbrace{-2x}_{A(x)}y + \underbrace{x}_{B(x)}$$

$A(x) = -2x$

$B(x) = x$

$$y(x) = e^{\int A(x)dx}\left(\int B(x)\, e^{-\int A(x)\,dx} + C\right)$$

$$\int A(x)\,dx = -\int 2x\,dx = -x^2 + C'$$

without loss of generality $C' = 0$

$$\int A(x)\,dx = -x^2$$

$$\int B(x)\, e^{-\int A(x)dx} = \int x\, e^{x^2} dx = \frac{1}{2} e^{x^2} + C''$$

we put $C'' = 0$ without loss of generality.

$$y(x) = e^{-x^2}\left(\frac{1}{2} e^{x^2} + C\right) = \frac{1}{2} + C e^{-x^2}$$