Brief summary of last week.

We have covered different types of ODEs.

$$\frac{dy}{dx} = f(x)$$ Review of calculus / simplest ODE

$$\frac{dy}{dx} = f(x)\,g(y)$$ separable ODE

Reducible to separable

$$\frac{dy}{dx} = f(ax+by+c)$$ Type I

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$ Scale invariant ODE (Type II)

TODAY

Examples

$$\frac{dy}{dx} = (\sin y)\,e^{x}$$ separable!

$$\frac{dy}{dx} = \tanh x$$ simplest (calculus)

$$\frac{dy}{dx} = \frac{1}{5x-2y}$$ reducible to separable type I.

Scale invariant ODE

Scale invariant 1^{st}-order ODEs are of the type

$$\boxed{\frac{dy}{dx} = f\left(\frac{y}{x}\right)}$$

They are called scale-invariant because they are invariant under the transformation

$$\begin{aligned} x &\to kx \\ y &\to ky \end{aligned} \qquad \text{where } k \in \mathbb{R}$$

Given $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$ under the scale transformation we have

LHS: $$\frac{dy}{dx} \to \frac{d(ky)}{d(kx)} = \frac{\cancel{k}dy}{\cancel{k}dx} = \frac{dy}{dx}$$ invariant ✓

RHS: $$f\left(\frac{y}{x}\right) \to f\left(\frac{\cancel{k}y}{\cancel{k}x}\right) = f\left(\frac{y}{x}\right)$$ invariant ✓

Solution of a scale-invariant 1^{st}-order ODE

These ODEs are separable upon a change of variable
(reducible to separable - Type II)

① We will denote $z = \frac{y}{x} \Rightarrow y = z \cdot x$ where $z = z(x)$

We observe that $y' = f\left(\frac{y}{x}\right) = f(z)$ (1)

② We differentiate $y(x) = z(x) \cdot x$ with respect to x

$y' = z + z' \cdot x \underset{\text{using (1)}}{=} f(z) \Rightarrow \cancel{z'x}\ z'x + z = f(z)$

$$\boxed{\begin{aligned} z' &= \frac{f(z) - z}{x} \\ \frac{dz}{dx} &= \frac{f(z) - z}{x} \end{aligned}}$$ separable!

Of the type

$\frac{dz}{dx} = g(z)\tilde{f}(x)$

where $g(z) = f(z) - z$
$\tilde{f}(x) = \frac{1}{x}$

③ We solve $z' = \frac{f(z) - z}{x}$ by separation of variables finding $z = z(x)$.

④ We solve for $y(x)$ by using

$$\boxed{y(x) = z(x) \cdot x}$$

□

<u>Example</u> $xy' = y - xe^{y/x}$

How to recognize this is a scale-invariant ODE?

a) Rearrange

b) Show the ODE is invariant under scaling

$x \to kx$

$y \to ky$

a) $y' = \frac{y}{x} - \frac{\cancel{x}}{\cancel{x}} e^{y/x} = \frac{y}{x} - e^{y/x} = f\left(\frac{y}{x}\right)$ ✓

b) Invariant under scale transformation $\begin{cases} x \to kx \\ y \to ky \end{cases}$

ODE: $xy' = y - xe^{y/x}$

$\frac{y}{x} \to \frac{y}{x}$

$y' \to y'$

Note that y'' is NOT scale invariant

$y'' = \frac{dy'}{dx} \to \frac{dy'}{dkx} = \frac{1}{k}\frac{dy'}{dx} = \frac{1}{k} y''$

LHS: $xy' \to (kx)y' = \boxed{k}(xy')$

RHS: $y - xe^{y/x} \to (ky) - (kx)e^{y/x} = \boxed{k}\left(y - xe^{y/x}\right)$

$\cancel{k}(xy') = \cancel{k}\left(y - xe^{y/x}\right)$ the ODE is scale invariant.

Solution of

$$xy' = y - x\, e^{y/x}$$

We write it in normal form

$$\boxed{y' = \frac{y}{x} - e^{y/x} = f\left(\frac{y}{x}\right)}$$

① We put $z = \frac{y}{x}$ $\Rightarrow y = z \cdot x$ $\quad y' = f(z) = z - e^{z}$ (*)

② $y' = z' \cdot x + z = f(z) = z - e^{+z}$

$$z' = \frac{f(z) - z}{x} = \frac{\cancel{z} - e^{z} - \cancel{z}}{x} \Rightarrow \boxed{z' = -\frac{e^{z}}{x}}$$ Separable

③ Solve $\frac{dz}{dx} = -\frac{e^{z}}{x}$ by separation of variables.

$$\int \frac{dz}{-e^{z}} = \int \frac{1}{x}\, dx + C'$$

LHS: $H(z) = -\int \frac{dz}{e^{z}} = -\int e^{-z}\, dz = e^{-z}$

RHS: $F(x) = \int \frac{1}{x}\, dx = \ln|x|$

Implicit solution for $z = z(x)$

$$\boxed{H(z) = F(x) + C'} \Rightarrow \boxed{e^{-z} = \ln|x| + C'}$$

Explicit solution for $z = z(x)$

$$\Rightarrow -z = \ln\left(\ln|x| + C'\right) \Rightarrow \boxed{z = -\ln\left(\ln|x| + C\right)}$$

(4) Solve for $y(x) = z(x) \cdot x$

$$\boxed{y(x) = -x \ln\left(\ln|x| + C'\right)}$$

Explicit solution for $y = y(x)$

$\square$