Solution of 1^{st}-order separable ODE (More formally)

We consider the separable ODE

$$\boxed{\frac{dy}{dx} = f(x)\, g(y)} \quad (2)$$

and an interval $y \in (A,B)$ in the domain of $g(y)$ such that

$$g(y) \neq 0$$

In this domain the ODE (2) has <u>implicit solution</u>

$$\boxed{H(y) = F(x) + C}$$

where $H(y) = \int \frac{1}{g(y)}\, dy$ & $F(x) = \int f(x)\, dx$

<u>Proof</u> $H(y) = \int \frac{1}{g(y)}\, dy \Rightarrow \frac{d}{dy} H(y) = \frac{1}{g(y)}$

We want to show that $H(y(x))$ is the antiderivative of f.

$F(x) = C + F(x) = \int f(x)\, dx$ $\qquad$ $\frac{d}{dx} F(x) = f(x)$

$\frac{d}{dx} F'(x) = f(x)$

If $\boxed{H(y(x)) = F(x) + C'} = F'(x)$

$$\iff \frac{d\,H(y(x))}{dx} = f(x) \qquad (*)$$

Let us consider $\dfrac{d}{dx} H(y(x)) = \dfrac{d}{dy} H(y) \cdot \dfrac{dy}{dx}$

But $\dfrac{d}{dy} H(y) = \dfrac{1}{g(y)}$ $\qquad \dfrac{dy}{dx} = f(x)\, g(y)$

$$\frac{d}{dx} H(y(x)) = \frac{1}{\cancel{g(y)}} f(x) \cancel{g(y)} = f(x)$$

$$\boxed{H(y(x)) = F(x) + C'}$$

Implicit solution

$\square$

If $H(y) = u$ admits several explicit expressions for its inverse functions the we need to consider all

inverse functions

$$y = H^{-1}(u)$$

$$\boxed{y = H^{-1}(F(x) + C)}$$

Explicit solution

Example $\quad \dfrac{dy}{dx} = \dfrac{x}{y} = f(x)\,g(y) \qquad f(x) = x \qquad g(y) = \dfrac{1}{y}$

$$\boxed{H(y) = F(x) + C} \quad \text{where}$$

$$H(y) = \int \frac{dy}{g(y)} = \int \frac{1}{\frac{1}{y}}\,dy = \int y\,dy = \frac{1}{2}y^2$$

$$F(x) = \int x\,dx = \int f(x)\,dx = \frac{1}{2}x^2$$

$$\boxed{\frac{1}{2}y^2 = \frac{1}{2}x^2 + C} \quad \text{Implicit solution}$$

$\cancel{y = H(u)}$ $\quad u = H(y) \qquad\qquad u = \dfrac{1}{2}y^2$

$$2u = y^2 \qquad y = \pm\sqrt{2u}$$

$$y = \begin{cases} \sqrt{2u} \\ -\sqrt{2u} \end{cases}$$

$$y = \begin{cases} \sqrt{2\left(\frac{1}{2}x^2 + C\right)} \\ -\sqrt{2\left(\frac{1}{2}x^2 + C\right)} \end{cases} \qquad \boxed{y = \begin{cases} \sqrt{x^2 + 2C} \\ -\sqrt{x^2 + 2C} \end{cases}} \quad \text{Explicit Solution}$$

$\square$

Reducible to separable 1st-order ODE

We consider the ODE of the type

$$\frac{dy}{dx} = f(ax + by + c)$$

where $a, b, c \in \mathbb{R}$ are constant parameters.

These equations are NOT separable but they are REDUCIBLE TO SEPARABLE

Solution of reducible to separable ODEs.

Step 1 Introduce the variable z

$$z = ax + by + c$$

$$f(z) = f(ax + by + c) \Rightarrow \frac{dy}{dx} = f(z)$$

$$\Rightarrow y = \frac{z - ax - c}{b}$$

Step 2 Observe that the ODE for $z(x)$ is separable

$$\frac{dz}{dx} = \frac{d}{dx}(ax + by + c) = a + b\underbrace{\frac{dy}{dx}}_{= f(z)} = a + bf(z)$$

$$\boxed{\frac{dz}{dx} = a + bf(z)}$$ is separable.

$**$

Step 3 Solve (**) by separation of variables. $z(x)$

Step 4 Set $$y(x) = \frac{z(x) - a x - c}{b}$$ $\square$

Example $$y' = e^{-(3x+y)} - 3$$

Is reducible to separable $y' = f(ax+by+c)$

$$\boxed{\begin{aligned} a &= 3 \\ b &= 1 \\ c &= 0 \end{aligned}}$$

Step 1 $z = ax + by + c$ $\qquad z = 3x + y$

$\Rightarrow y = z - 3x$ $\qquad \frac{dy}{dx} = y' = e^{-z} - 3 = f(z)$

Step 2 $$\frac{dz}{dx} = \frac{d}{dx}(3x+y) = 3 + \frac{dy}{dx} = \cancel{3} + (e^{-z} - \cancel{3})$$

$$\frac{dz}{dx} = e^{-z}$$ Separable ODE.

Step 3 Solve for $z = z(x)$

$$\frac{dz}{dx} = e^{-z} \Rightarrow \int \frac{dz}{e^{-z}} = \int dx + C'$$

LHS: $H(z) = \int e^z \, dz = e^z$

RHS: $F(x) = \int dx = x$

Implicit solution for $z(x)$ $\quad H(z) = F(x) + C'$

$$\boxed{e^z = x + C'}$$

with C' arbitrary constant.

$u = H(z) = e^z \qquad z = \ln u$

$$\boxed{z = \ln(x + C')}$$

Explicit solution for $x + C' > 0$

<u>Step 4</u>

$y = z - 3x$

$$\boxed{y = \ln(x + C') - 3x \qquad \text{for } x + C' > 0}$$