The simplest ODE

$$\boxed{\frac{dy}{dx} = f(x)} \qquad (1)$$

Equation (1) has solution (see calculus)

$$y(x) = \int f(x)\,dx + C \qquad \text{General solution}$$

where C is an arbitrary costant

Initial value problem (IVP)

1) ODE e.g. $\frac{dy}{dx} = f(x)$

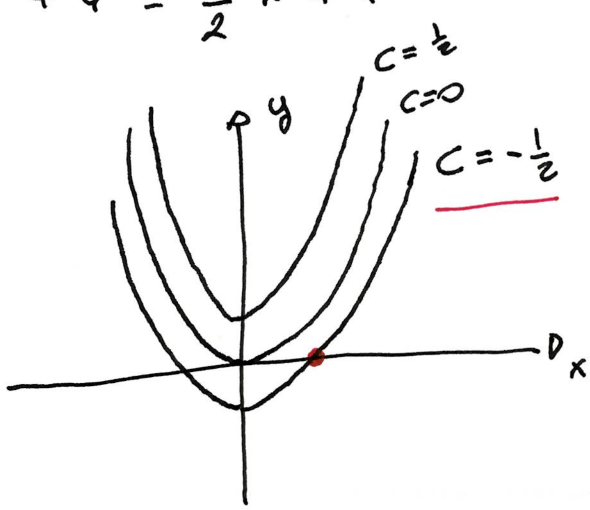
2) Initial condition $y(x_0) = y_0$
I.C.

Example $\frac{dy}{dx} = x$ & I.C. $y(1)=0$

General solution $y(x) = \int x\,dx + C = \frac{1}{2}x^2 + C$

I.C. $y(x_0) = y_0$

$$\boxed{\begin{array}{l} x_0 = 1 \\ y_0 = 0 \end{array}} \qquad (1,0)$$



The initial condition determines the value of the constant C

$y = \frac{1}{2}x^2 + C'$ General solution

Impose the I.C. $\Rightarrow$ fix C'

$0 = y(1) = \frac{1}{2}\cdot(1)^2 + C'$ $\Rightarrow$ $\frac{1}{2} + C' = 0$

$\Rightarrow \boxed{C = -\frac{1}{2}}$

$$\boxed{y(x) = \frac{1}{2}x^2 - \frac{1}{2}}$$

Solution to the IVP

ODE + I.C.

The general solution is given in terms of an arbitrary constant

This constant can be fixed by imposing the initial conditions

1^{st}-order separable ODE

A 1^{st}-order ODE in mormal form

$$y' = \tilde{f}(x,y) \text{ is } \textbf{separable} \text{ if}$$

$$\tilde{f}(x,y) = f(x)\, g(y) \quad \text{i.e}$$

$$(2) \quad \boxed{\begin{array}{l} y' = f(x)\, g(y) \\ \text{or equivalently} \\ \dfrac{dy}{dx} = f(x)\, g(y) \end{array}}$$

Separable 1^{st}-order ODE

<u>Examples</u>

	$f(x)$	$g(y)$	Seperable
$y' = e^x y$	e^x	y	Yes!
$y' = \dfrac{e^x y}{y+1} = e^x \cdot \dfrac{y}{y+1}$	e^x	$\dfrac{y}{y+1}$	Yes!
$y' = e^x \cdot 1$	e^x	1	Yes!
$y' = e^{x+3y} = e^x \cdot e^{3y}$	e^x	e^{3y}	Yes!
$y' = x + 3y$			No!
$y' = (x+3y)^{2/3}$			No!

1^{st} - order separable ODE : formal solution

Let us proceed formally (more & rigorous derivation in lesson 3 Week 1)

$$\boxed{\frac{dy}{dx} = f(x)\, g(y)}$$

Step 1 Identify $f(x)$ and $g(y)$

Separate the variables

$$\frac{dy}{g(y)} = f(x)\, dx \qquad g(y) \neq 0$$

Step 2 Integrate both sides

$$\int \frac{dy}{g(y)} = \int f(x)\, dx + C'$$

$H(y)$ the antiderivative of $[g(y)]^{-1}$ $\qquad H(y) = \int \frac{dy}{g(y)}$

$F(x)$ " " " $f(x)$ $\qquad F(x) = \int f(x)\, dx$

$$\boxed{H(y) = F(x) + C'}$$ Implicit solution

Step 3 If the inverse of $H(y) = u$ has an explicit expression

$y = H^{-1}(u)$

$$y = H^{-1}\left(F(x) + C'\right) \qquad (3)$$ Explicit solution

Observations

1. Note that there might be more than one inverse function

$H^{-1}(u)$

ex. $H(y) = u$ $\quad y^2 = u \quad \Rightarrow y = \pm\sqrt{u}$

$$\Rightarrow \begin{cases} y = \sqrt{u} \\ y = -\sqrt{u} \end{cases}$$

You need to list all solutions of type (3)

2. Not always $H(y)$ can be inverted. In that case you can only provide the implicit solution.

Example $\quad y' = xy^2$

Step 1 $\quad f(x) = x \quad g(y) = y^2$

$$\frac{dy}{dx} = xy^2 \quad \Rightarrow \quad \frac{dy}{y^2} = x\,dx$$

Step 2

$$\int \frac{dy}{y^2} = \int x\,dx + c'$$

LHS: $\quad H(y) = \int \frac{1}{y^2}\,dy = \int y^{-2}\,dy = -\frac{1}{y}$

RHS: $\quad F(x) = \int x\,dx = \frac{1}{2}x^2$

$$H(y) = F(x) + C'$$

$$\boxed{-\frac{1}{y} = \frac{1}{2}x^2 + C}$$ Implicit solution

<u>Step 3</u>

$$H(y) = -\frac{1}{y} = u \qquad y = H^{-1}(u) = -\frac{1}{u}$$

$$-\frac{1}{y} = u \qquad -\frac{1}{u} = y$$

$$y = H^{-1}(F(x) + C')$$

$$\boxed{y = -\frac{1}{\frac{1}{2}x^2 + C'}}$$ Explicit solution

$$y' = x y^2 \qquad \boxed{y(x) = 0}$$ Other solution.

Seperable 1^{st}-order ODEs: constant solutions

Given a 1^{st}- order separable ODE

$$\frac{dy}{dx} = f(x)\,g(y) \qquad (2)$$

Let $y_0, y_1 \ldots y_k$ be the roots $g(y)$

therefore $g(y_r) = 0 \quad \forall r \in \{1, 2, \ldots k\}$

Then the constant function $y(x) = y_r$ is a solution of (2)

Example

$g(y) = (y-1)(y-2)$

$f(x) = 1$

$$\frac{dy}{dx} = (y-1)(y-2) \quad *$$

Roots of $g(y)$ are $y_1 = 1 \quad y_2 = 2$

$\Rightarrow \bar{y}(x) = 1 \qquad \bar{y}(x) = 2$ are solution of (*)