Practice question - Formative Assessment 1

A.1) $y' = 3x^2 + 2x + 3$ & $y(0) = 1$

$y' = f(x)$ $y(x_0) = y_0$

$\boxed{\begin{array}{l} x_0 = 0 \\ y_0 = 1 \end{array}}$ $(0,1)$

$y(x) = \int f(x)\,dx + C$ General solution

$$y(x) = \int (3x^2 + 2x + 3)\,dx + C = x^3 + x^2 + 3x + C$$

C arbitrary constant.

$$\boxed{y(x) = x^3 + x^2 + 3x + C}$$ General solution

$1 = y(0) = C \Rightarrow C = 1$

$$\boxed{y(x) = x^3 + x^2 + 3x + \cancel{C}\, 1}$$ Solution to the I.V.P.

A.3) $y' = (x+1)\, e^{-x^2-2x}$ & I.C. $y(0) = 1$

$y' = f(x)$

$y(x_0) = y_0$

$\boxed{\begin{array}{l} x_0 = 0 \\ y_0 = 1 \end{array}}$

$y(x) = \int f(x)\, dx + C$ General solution

where C is an arbitrary constant.

$y(x) = \int \underbrace{(x+1)}\, e^{-x^2-2x}\, \underbrace{dx} + C$

$u = -x^2 - 2x$

$du = (-2x-2)\, dx = -2(x+1)\, dx$ $(x+1)\,dx = -\frac{du}{2}$

$y(x) = \int e^u \left(-\frac{1}{2}\right) du = -\frac{1}{2}\int e^u\, du = -\frac{1}{2} e^u + C$

$\boxed{y(x) = -\frac{1}{2} e^{-x^2-2x} + C}$ General solution

$1 = y(0) = -\frac{1}{2} e^0 + C = -\frac{1}{2} + C \Rightarrow C = 1 + \frac{1}{2} = \frac{3}{2}$

$\boxed{y(x) = -\frac{1}{2} e^{-x^2-2x} + \frac{3}{2}}$ Solution to the I.V.P.

Exponential growth / decay.

Applications
- Ecology - growth of a population of rabits.
- Finance - growth of a saving due to interest rate compounded continuously
- Lightbulb problem - Failure of lightbulbs in time (decay)
- Radioactive decay - How the radioactive material decays in time.

All these applications follow the same ODE:

$\boxed{y' = \lambda y}$ & $y(x_0) = y_0$

Example: $\boxed{y' = \lambda y}$ & $\boxed{y(0) = 2}$ I.C. $\boxed{\begin{matrix} x_0 = 0 \\ y_0 = 2 \end{matrix}}$

① Separable? $y' = f(x)\,g(y)$ Yes! $f(x) = \lambda$ $g(y) = y$

② $\frac{dy}{dx} = \lambda y \Rightarrow \int \frac{dy}{y} = \int \lambda \, dx + C$

③ LHS $H(y) = \int \frac{dy}{y} = \ln|y|$

RHS $F(x) = \int \lambda \, dx = \lambda x$

$H(y) = F(x) + \tilde{C}$ Implicit solution

$\ln|y| = \lambda x + C$ where $\tilde{C}$ is an arbitrary constant

$H(y) = u \quad \Rightarrow \quad H(y) = \ln|y| = u$

$y = \pm e^{u} = H^{-1}(u)$

$$y = \pm e^{\lambda x + C} = \pm e^{\lambda x} \cdot e^{C} = \underbrace{\boxed{\pm e^{C}}}_{=D} e^{\lambda x} = D e^{\lambda x}$$

$y(x) = D e^{\lambda x}$ where D is an arbitrary constant.

Costant solutions? $g(y) = y = 0$ for $y = 0$

$y(x) = 0$ is a constant solution.

General solutions $\boxed{\begin{array}{l} y(x) = D e^{\lambda x} \\ y(x) = 0 \end{array}}$ where D is an arbitrary constant.

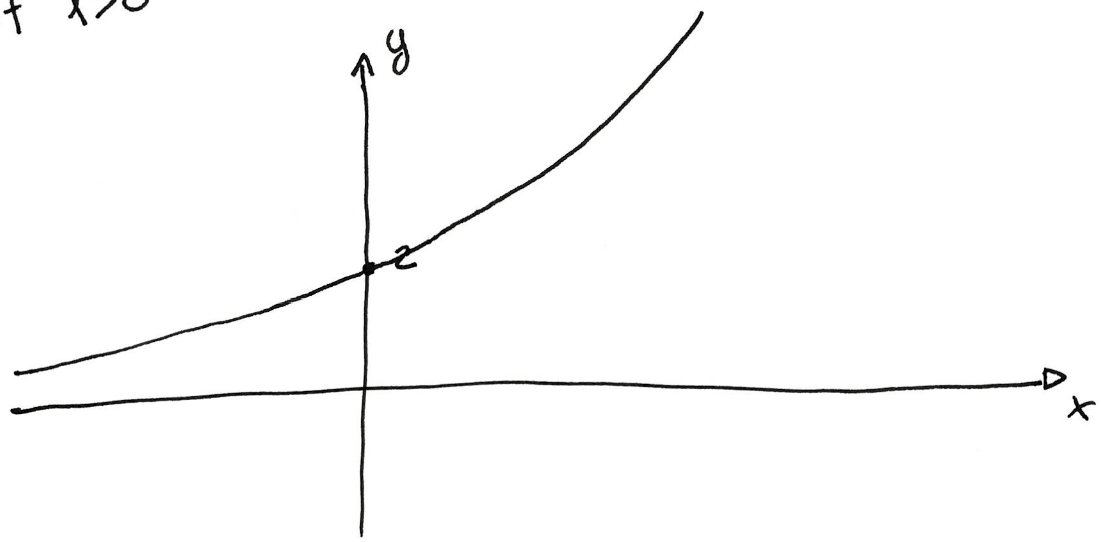
I.C.

$2 = y(0) \neq 0$ $\quad 2 = y(0) = D e^{\lambda \cdot 0} = D e^{0} = D$

$\Rightarrow D = 2$

$\boxed{y(x) = 2 e^{\lambda x}}$ Solution to the IVP.

If $l>0$



If $l<0$