LTCC, Measure-Theoretic Probability: Examination AY 2021/22

You may refer without proof to results from the course (theorems, examples, etc.).

Q1

(i) Show that if events $A_1, A_2, \ldots$ are independent then

$$\mathbb{P}\left(\bigcap_{n=1}^{\infty} A_n\right) = \prod_{n=1}^{\infty} \mathbb{P}(A_n).$$

(ii) Show that for any random variable ξ there exists a numerical sequence $c_1, c_2, \ldots$ such that, as $n \to \infty$,

$$c_n \xi \xrightarrow{\mathbb{P}} 0.$$

(iii) Prove that for any sequence of random variables $\xi_1, \xi_2, \ldots$ there exist constants $c_1, c_2, \ldots$ such that the series $\sum_{n=1}^{\infty} c_n \xi_n$ converges almost surely.

(iv) Let $\xi_1, \xi_2, \ldots$ be i.i.d. with some probability mass function $p(i)$ on $\{0, 1, \ldots, 9\}$, and let $\eta_1, \eta_2, \ldots$ be i.i.d., with some probability mass function $q(i)$ on $\{0, 1, \ldots, 9\}$.

Consider the random variables

$$X = \sum_{n=1}^{\infty} \frac{\xi_n}{10^n}, \quad Y = \sum_{n=1}^{\infty} \frac{\eta_n}{10^n}.$$

Let μ be the probability distribution of X, and ν be the probability distribution of Y.

Show that μ and ν are not absolutely continuous relative to one another (i.e. *mutually singular*), unless $p(i) = q(i), i \in \{0, 1, \cdots, 9\}$.

Q2

(i) Give example of two random processes $(X(t), t \geq 0)$ and $(Y(t), t \geq 0)$ which have different distributions but satisfy $X(t) = Y(t)$ a.s. for each fixed t.

(ii) Let $X, X_1, X_2, \ldots$ be discrete random variables with values in $\mathbb{Z}$. Show that the weak convergence $X_n \Rightarrow X$ holds if and only if $\lim_{n\to\infty} \mathbb{P}(X_n = k) = \mathbb{P}(X = k),\ k \in \mathbb{Z}$.

(iii) Suppose X_n is a $[0, 1]$-valued random variable with density

$$f_n(x) = 1 - \cos(2\pi n x), \quad x \in [0, 1].$$

Determine a weak limit of the X_n's as $n \to \infty$. Do the densities f_n converge pointwise?

(iv) Let $X_1, X_2, \ldots$ be i.i.d. random variables with density f, and let g be another density function. The likelihood ratio for a sample of size n is defined as

$$L_n = \frac{\prod_{i=0}^{n} g(X_i)}{\prod_{i=0}^{n} f(X_i)}.$$

Show that $L_1, L_2, \ldots$ is a martingale (adapted to the natural filtration generated by $X_1, X_2, \ldots$).

Q3 Let $(B(t),\ t \geq 0)$ be the Brownian motion (BM) with the natural filtration $(\mathcal{F}_t^B, t \geq 0)$. Consider

$$N := \{t \geq 0 : B(t) = 0\},$$

the set of zeroes of the BM.
[In this question, if the probability of event is not specified, it is meant that the event occurs with probability one. Furthermore, you may rely on the fact that the process

$$W(t) := B(t + \tau) - B(\tau),\ t \geq 0,$$

is again a BM, for every finite stopping time τ adapted to $(\mathcal{F}_t^B, t \geq 0)$.]

(i) Is N a Borel set? Is it closed? Justify your answer.

(ii) Show that N has zero Lebesgue measure. [Hint: use Fubini's theorem.]

(iii) For $x \neq 0$, let $\tau_x = \inf\{t : B(t) > x\}$. Show that $\tau_x < \infty$.

(iv) Show that N has infinite cardinality.

(v) Argue that N has points isolated from the left, and points isolated from the right.

(vi) Verify that the process $(tB(1/t),\ t \geq 0)$ is a BM and use this fact to show that $t = 0$ is not an isolated point of N.

(vii) Let $D_t := \inf([t, \infty) \cap N)$. Show that

$$N_D := \bigcup_{t \notin N} \{D_t\}$$

(union of one-point sets) is a countable set, that $N_D \subset N$, and that each $t \in N_D$ is not an isolated point of N.

(viii) Prove that N has no isolated points.

(ix) The Laplace transform of τ_x in part (iii) is

$$\mathbb{E}e^{-\lambda \tau_x} = e^{-|x|\sqrt{2\lambda}}.$$

By using this and conditioning on $B(1)$, find the Laplace transform and the density of the random variable of $D_1 - 1$.

(x) Derive from (ix) that 'the last zero' $\max(N \cap [0, 1])$ follows the arcsine distribution with density function

$$f(t) = \frac{1}{\pi\sqrt{t(1-t)}},\quad t \in [0, 1].$$