Measure-Theoretic Probability 2021/22

Q1 (i) The events $B_n = A_1 \cap \dots \cap A_n$ satisfy $B_1 \supset B_2 \supset \dots$

hence $\mathbb{P}(\bigcap_{n=1}^{\infty} A_n) = \mathbb{P}(\bigcap_{n=1}^{\infty} B_n) = \lim_{n\to\infty} \mathbb{P}(B_n)$

$= \lim_{n\to\infty} \mathbb{P}(\bigcap_{j=1}^{n} A_j) = \lim_{n\to\infty} \prod_{j=1}^{n} \mathbb{P}(A_j)$ (by indep.)

$= \prod_{j=1}^{\infty} \mathbb{P}(A_j)$

(ii) Since $\lim_{n\to\infty} \mathbb{P}(|\xi| > n\varepsilon) = 0 \quad \forall \varepsilon > 0$

we get $\lim_{n\to\infty} \mathbb{P}(\frac{1}{n}|\xi| > \varepsilon) = 0$

So $c_n = 1/n$ is a sought sequence

(iii) According to (ii), for each n it is possible to choose c_n to achieve

$$\mathbb{P}(c_n|\xi_n| > \tfrac{1}{n^2}) < \tfrac{1}{n^2}.$$

Appealing to the Borel-Cantelli Lemma

$\mathbb{P}(c_n|\xi_n| > 1/n^2 \text{ i.o.}) = 0$ because $\sum 1/n^2 < \infty$.

But then $c_n|\xi_n| \le \frac{1}{n^2}$ for all n (but finitely many) almost surely.

In this event $\sum_n c_n|\xi_n| < \infty$, so

$$\mathbb{P}(\sum_n |\xi_n| c_n < \infty) = 1.$$

Q 1 (IV) Note that ξ_n is the n-th digit of X

η_n — " — Y

Suppose for some i $\quad p(i) \neq q(i)$

By the Law of Large Numbers

$$\frac{\#\{n : \xi_n = i\}}{n} \to p(i) \text{ a.s.} \qquad (*)$$

$$\frac{\#\{n : \eta_n = i\}}{n} \to q(i) \text{ a.s.}$$

Let A be the event, meaning that the frequency of i in decimal expansion of X equals $p(i)$

B — " — Y equals $q(i)$

Then $\mu(A) = 1$, $\nu(A) = 0$

$\mu(B) = 0$, ~~$\mu(B) = 1$~~ $\nu(B) = 1$

$\Rightarrow$ μ, ν are mutually singular

Q2 (i) Let $(X(t), t \geq 0)$ be a Brownian motion,
U independent uniform-$[0,1]$ random variable.
Set $Y(t) = X(t) + 1_{U}$

Then $X(t) = Y(t)$ for all $t \neq U$.

Since $\mathbb{P}(U = t) = 0$ for each t.

$\Rightarrow X(t) = Y(t)$ a.s., so also $X(t) \overset{d}{=} Y(t)$

However, $\mathbb{P}\left(\max_{t \in [0,1]} X(t) < \max_{t \in [0,1]} Y(t)\right) > 0$

Thus the processes have different distributions.

(Note: restricting the process to $[0,1]$, the distribution is a probability measure on the $D[0,1]$ space).

(ii) For each k choose any bounded continuous function with $f_k(k) = 1$, $f_k(j) = 0$ for $j \neq k$

Then $X_n \Rightarrow X$ implies $\mathbb{E} f_k(X_n) \to \mathbb{E} f_k(X)$

$$\underset{\mathbb{P}(X_n = k)}{\underbrace{\mathbb{E} f_k(X_n)}} \to \underset{\mathbb{P}(X = k)}{\underbrace{\mathbb{E} f_k(X)}}$$

Conversely, if $\mathbb{P}(X_n = k) \to \mathbb{P}(X = k)$

then $\mathbb{E} f(X_n) = \sum_k f(k) \mathbb{P}(X_n = k) \to \sum_k f(k) \mathbb{P}(X = k)$

by bounded convergence.

Q2 (iii) The cumulative distribution function is

$$F_n(x) := \int_0^x f_n(y)\,dy = x - \frac{\sin(n\pi x)}{n\pi}$$

Letting $n \to \infty$, $F_n(x) \to x$ for all $x \in [0,1]$

This corresponds to the uniform distribution, which is the weak limit.

But f_n's do not converge themself.

E.g. $\cos(2\pi n x)$ does not converge for irrational x.

(IV) Let $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$

$$\mathbb{E}(L_{n+1} \mid \mathcal{F}_n) = \mathbb{E}\left(\left(\underbrace{\frac{\prod_{i=0}^{n} g(X_i)}{\prod_{i=0}^{n} f(X_i)}}_{\mathcal{F}_n\text{-measurable}} \frac{g(X_{n+1})}{f(X_{n+1})} \,\middle|\, \mathcal{F}_n\right)\right)$$

$$= L_n \mathbb{E}\left[\frac{g(X_{n+1})}{f(X_{n+1})} \,\middle|\, \mathcal{F}_n\right] = \text{by independence}$$

$$L_n \mathbb{E}\left[\frac{g(X_{n+1})}{f(X_{n+1})}\right] = L_n \int_{-\infty}^{\infty} \frac{g(x)}{f(x)} f(x)\,dx =$$

$$= L_n \int_{-\infty}^{\infty} g(x)\,dx = L_n$$

$\Rightarrow (L_n, n \geq 0)$ is a martingale)

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Q3 (i) Yes, since $(B(t), t \geq 0)$ has continuous paths, N is closed hence $N \in \mathcal{B}(\mathbb{R})$.

(ii) λ - Lebesgue measure

$$\lambda(N) = \int_0^\infty 1_{\{t \in N\}} \, dt$$

$$\mathbb{E}\lambda(N) = \mathbb{E}\int_0^\infty 1_{\{t \in N\}} \, dt$$

$$= \int_0^\infty \mathbb{P}(t \in N) dt = 0$$

because $t \in N \Leftrightarrow B(t) = 0$, and $B(t)$ has continuous distribution.

(iii) note that $B(t) > x \Rightarrow \tau < t$, so $\tau < \infty$

$$\mathbb{P}(B(t) > x) = \mathbb{P}\left(B(1) > \frac{x}{\sqrt{t}}\right) \to 1 \quad \text{as } t \to \infty.$$

$\Rightarrow \tau_x < \infty$ a.s.

(iv) This can be concluded from (iii)

$\tau_1 < \infty$. $B(t + \tau_1) - B(\tau_1)$ is again a BM ~~hence with positive~~ hence will reach -1 a.s. at some time $\tilde{\sigma}$

By continuity, between τ_1 and σ there is t with $B(t) = 0$.

After time $\tau_1 + \sigma$ the process is a BM hence will have another zero, etc by induction.

(V) For each t, $\mathbb{P}(B(t) \neq 0) = 1$, thus there exists an interval (G_t, D_t) such that $t \in (G_t, D_t)$ and

$B(G_t) = B(D_t) = 0$; $G_t \in N$ is isolated from the right, and $D_t \in N$ is isolated from the left.

(VI) $W(t) = t\,B(1/t)$ is Gaussian, with continuous paths; for $t > 0$ this is obvious while for $t = 0$ continuity follows from $B(t)/t \to 0$ a.s for $t \to \infty$.

$\mathbb{E}W(t) = 0$, $\mathrm{Cov}(W(s), W(t)) = st \min(1/t, 1/s)$

$$= \min(s, t)$$

$\Rightarrow (W(t), t \geq 0)$ is a BM

If $W(t) = 0$ then $B(1/t) = 0$ (and conversely)

Since B has $t_1, t_2, \dots \to \infty$ with $B(t_n) = 0$

we have $W(1/t_n) = 0$ so 0 is not isolated zero of W.

(vii) The intervals (G_t, D_t) either coincide for different t, or disjoint.

Hence $\{G_t, t \geq 0\}$ is a countable set

$\{D_t, t \geq 0\}$ is also a countable set.

The random variable D_t is a stopping time, therefor by the strong Markov property

$$B(s+D_t) - B(D_t) = B(s+D_t) \qquad s \geq 0$$

is a BM.

That $D_t \in N$ is not isolated follows from part (vi)

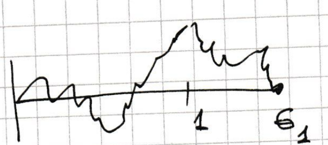
(viii) If t is isolated from the left, $t \in N$, then

$t = G_s$ for some rational s.

$$\mathbb{P}(t \in \{G_s, s>0\}, t \text{ isolated}) =$$

$$= \mathbb{P}(t \in \{G_s, s \in \mathbb{Q}\}, t \text{ isolated}) \leq$$

$$\sum_{s \in \mathbb{Q}} \mathbb{P}(G_s \text{ isolated}) = 0 \quad \text{by part (vii)}$$

(ix) 

Conditional on $B(1) = x$, $G_1 - 1$ has same distribution as τ_x. Thus the Laplace transform of $G_1 - 1$ is

$$E e^{-\lambda(G_1-1)} = \int_{-\infty}^{\infty} e^{-|x|\sqrt{2\lambda}} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx = e^{\lambda} \operatorname{Erfc}(\sqrt{\lambda})$$

$$= \frac{2}{\sqrt{\pi}} \int_{\sqrt{\lambda}}^{\infty} e^{-x^2} dx$$

The inverse Laplace transform is the density

$$g(t) = \frac{1}{\pi\sqrt{t}\,(1+t)}$$

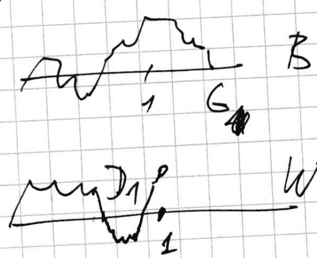
of $G_1 - 1$.

(x) We use that $W(t) = t\,B(1/t)$ is a BM, see part [illegible])

By this transform the first zero of B on $[1,\infty)$ becomes the last zero of W on $[0,1]$

$G_1 - 1$ for B

is mapped to $1 - D_1$



g is transformed into

$$f(t) = \frac{1}{\pi\sqrt{t(1-t)}}, \quad t \in [0,1]$$

Since this is symmetric with respect to

$$t \leftrightarrow 1-t$$

$D_1 \overset{d}{=} 1 - D_1$, so D_1 has the arcsine density function f.