Important information about the module and its assessment

Module

Weeks 1-3	1^{st}-order ODEs, IVP, Picard-Lindelöf theorem
Weeks 4-6	2^{nd}-order ODEs, BVP, Theorem of the Alternative.
Weeks 8-11	Systems of 1^{st}-order ODEs, Phase portraits.
Weeks 12	Revision week.

Assessment

Coursework 1	Quiz based	10% final mark
	Well done!	
Coursework 2	Quiz based	10% final mark
	Friday 8 Dec 12pm - Friday 15 Dec 12pm	

Final exam

80% of final mark

Handwritten - 4 Questions

Train with formative assessment + Past papers.

Mock exam

System of 1st-order ODEs

A system of 1st-order ODEs having t as independent variable and $y_1, y_2 \dots y_n$ as dependent variables is reads in normal form

$$\begin{cases} \dot{y}_1 = f_1(t, y_1, y_2, \dots y_n) \\ \dot{y}_2 = f_2(t, y_1, y_2, \dots y_n) \\ \vdots \\ \dot{y}_n = f_n(t, y_1, y_2, \dots y_n) \end{cases} \quad (1)$$

A system of ODEs having t as independent variable (such as (1)) is called a DYNAMICAL SYSTEM.

A dynamical system is AUTONOMOUS if all the functions $f_i(t, y_1, y_2 \dots y_n)$ are independent of time t

$$f_i(t, y_1 y_2 \dots y_n) = f(y_1, y_2 \dots y_n)$$

In this module we will consider only dynamical systems of 1^{st}-order ODEs with two dependent variables.

$$\dot{y}_1 = f_1(t, y_1, y_2) \qquad \text{or} \qquad \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} f_1(t, y_1, y_2) \\ f_2(t, y_1, y_2) \end{pmatrix}$$
$$\dot{y}_2 = f_2(t, y_1, y_2)$$

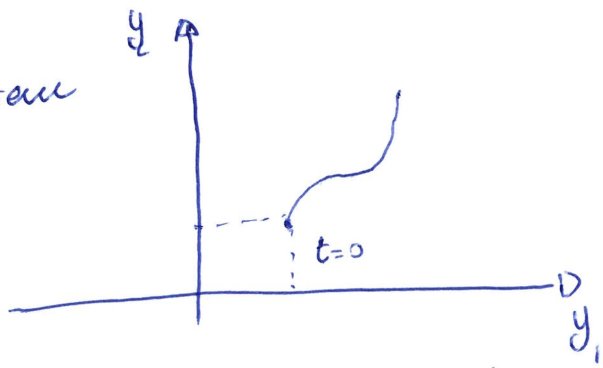
Example

$$\begin{cases} \dot{y}_1 = y_1 + e^{y_2} \\ \dot{y}_2 = \tan(y_1 + y_2) \end{cases}$$ <u>AUTONOMOUS</u>

$$\begin{cases} \dot{y}_1 = e^{t} + y_1^3 y_2 \\ \dot{y}_2 = \sin y_1 \end{cases}$$ <u>NON-AUTONOMOUS</u>

Phase space

Consider the autonomous dynamical system



$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} f_1(y_1, y_2) \\ f_2(y_1, y_2) \end{pmatrix}$$

A region of the two dimensional space $\mathbb{R}^2$ described by the coordinates $(y_1 y_2)$ where the functions $f_1(y_1, y_2)$ and $f_2(y_1, y_2)$ are well defined is called the PHASE SPACE of the system.

We will only consider dynamical system where the phase space is the (y_1, y_2) plane $\mathbb{R}^2$

Initial Value Problem for Autonomous Dynamical Systems.

The IVP for autonomous dynamical system comprises

System of ODEs:
$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} f_1(y_1, y_2) \\ f_2(y_1, y_2) \end{pmatrix}$$

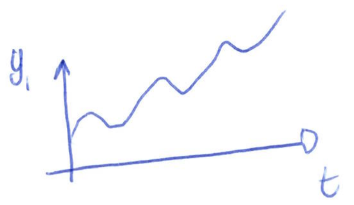
ICs:
$$\begin{cases} y_1(a) = b_1 \\ y_2(a) = b_2 \end{cases} \quad \text{at time } t = a$$

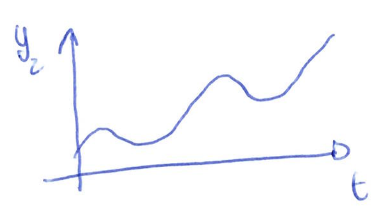
We will consider solutions with $t \in (-\infty, \infty)$ both future and past.

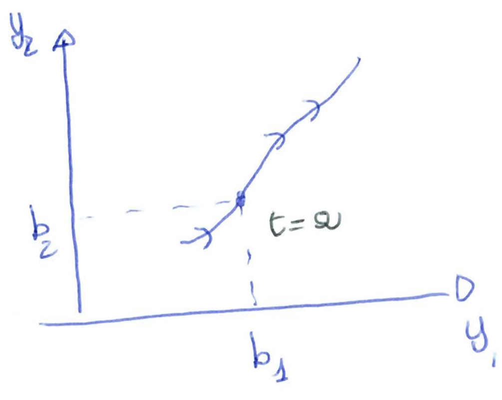
We will consider only dynamical systems for which f_1, f_2 and the partial derivatives $\dfrac{\partial f_i}{\partial y_j}$ $i \in \{1,2\}$, $j \in \{1,2\}$ are continuous in $\mathbb{R}^2$

In this hypotheses the Picard-Lindelöf theorem ensures the existence and uniqueness of the IVP.

$$\begin{cases} y_1 = y_1(t) \\ y_2 = y_2(t) \end{cases}$$







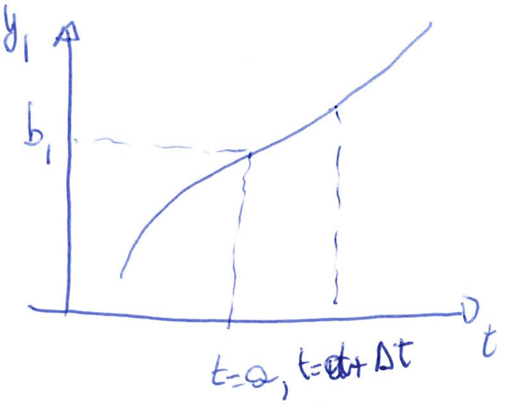
Trajectories of dynamical systems

Let us consider an autonomous dynamical system

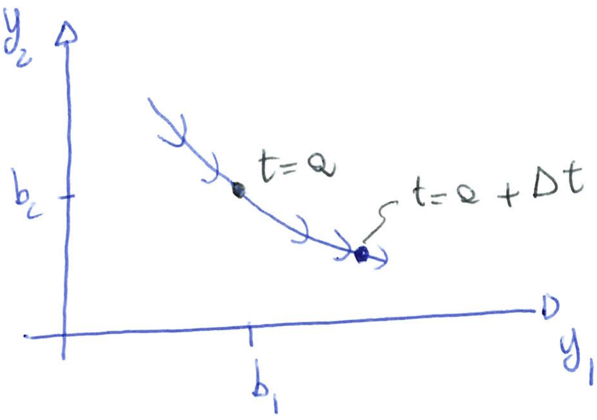
$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} f_1(y_1, y_2) \\ f_2(y_1, y_2) \end{pmatrix} \quad \text{and ICs.} \quad \begin{cases} y_1(0) = b_1 \\ y_2(0) = b_2 \end{cases}$$

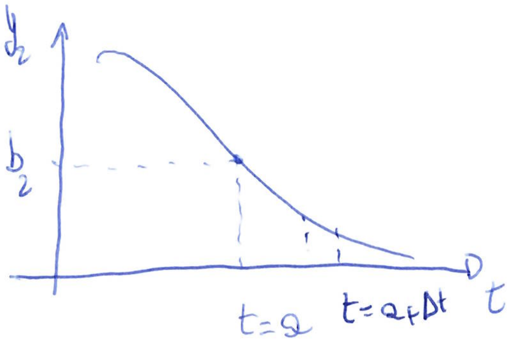
with solution

$$\begin{cases} y_1 = y_1(t) \\ y_2 = y_2(t) \end{cases} \qquad t \in (-\infty, \infty)$$



TRAJECTORY





In phase space (y_1, y_2) $y_1 = y_1(t)$ and $y_2 = y_2(t)$ describe a curve that is called TRAJECTORY

In other words $y_1 = y_1(t)$ and $y_2 = y_2(t)$ describe a parametrized curve in the phase space with parameter t.

This is the TRAJECTORY.

Example $\begin{cases} \dot{y}_1 = y_1 \\ \dot{y}_2 = 2y_2 \end{cases}$ IC: $\begin{cases} y_1(0) = 1 \\ y_2(0) = 1 \end{cases}$

Solving $\dot{y}_1 = y_1$ & $y_1(0) = 1$

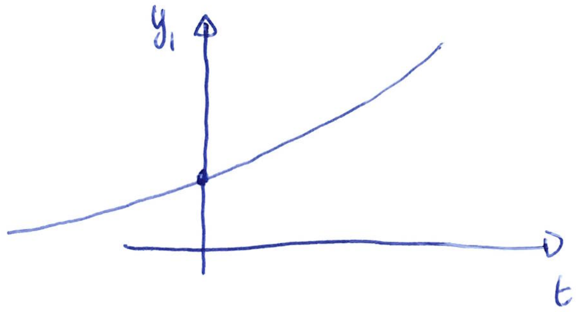
(check at home) $\boxed{y_1(t) = e^t}$

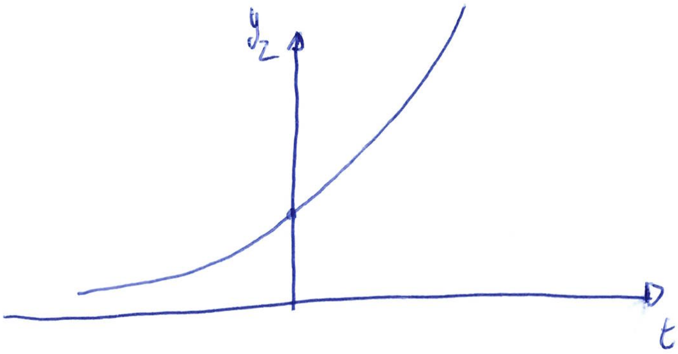
Solving $\dot{y}_2 = 2y_2$ & $y_2(0) = 1$

(check at home) $\boxed{y_2(t) = e^{2t}}$

Solution of the dynamical system

$\begin{cases} y_1 = e^t \\ y_2 = e^{2t} \end{cases}$





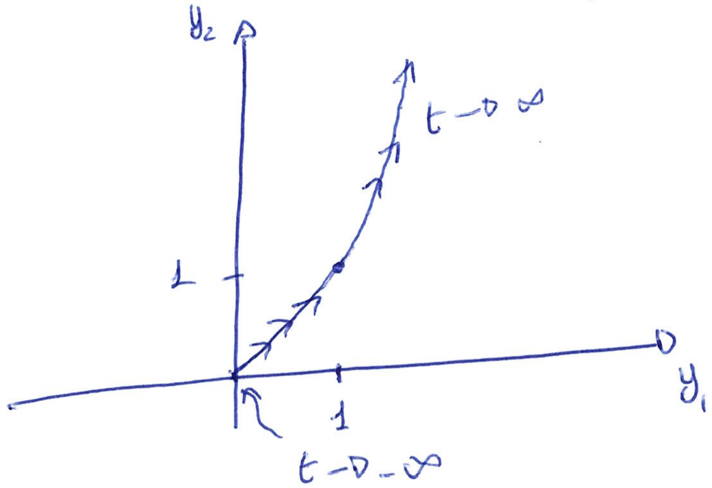
The trajectory is a parabola

$y_2 = y_1^2$

$y_1 = e^t$, $y_1 > 0$

$y_2 = (y_1)^2 = (e^t)^2 = e^{2t}$

$\boxed{y_2 = y_1^2}$



Trajectories - properties

$$\boxed{\text{Trajectories either completely coincide or they do not have any point in common}}$$

This is a consequence of the Picard - Lindelöf theorem, and it is valid under the hypotheses of this theorem.

If two different trajectories had a point in common, then using that point as an initial condition we would have two different solutions of the IVP. This will contradict the Picard - Lindelöf theorem. So this is excluded.

Equilibria of dynamical systems

The autonomous system

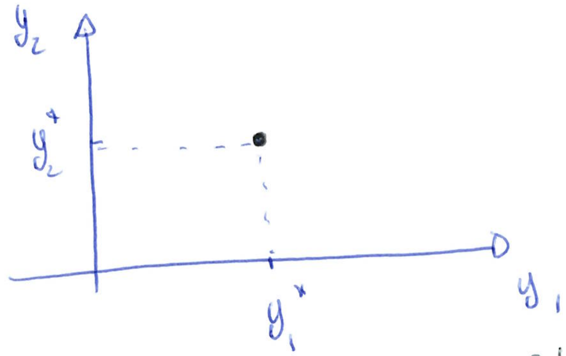
$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} f_1(y_1, y_2) \\ f_2(y_1, y_2) \end{pmatrix}$$

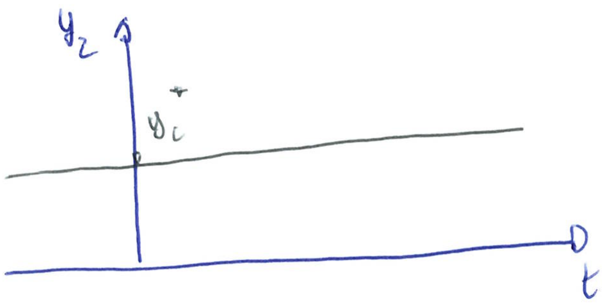
has an equilibrium point at $(y_1, y_2) = (y_1^*, y_2^*)$ if and only if

$$\begin{cases} \dot{y}_1 = f_1(y_1^*, y_2^*) = 0 \\ \dot{y}_2 = f_2(y_1^*, y_2^*) = 0 \end{cases}$$

From this definition it follows that an IVP with

I.C. $\begin{cases} y_1(a) = y_1^* \\ y_2(a) = y_2^* \end{cases}$ has solution $\begin{cases} y_1(t) = y_1^* \\ y_2(t) = y_2^* \end{cases}$





The trajectory of an equilibrium point is (y_1^*, y_2^*)

(y_1^*, y_2^*) is called <u>EQUILIBRIUM POINT</u>

Equilibria

A dynamical system can have many equilibria

Equilibrium - singular
Equilibria - plural

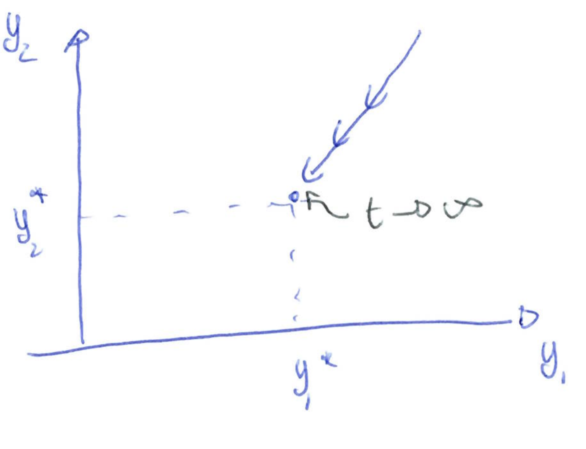
Equilibria are also called FIXED POINTS, STATIONARY POINTS or SINGULAR POINTS.

Any solution to an IVP with I.C. different from an equilibrium point can only reach an equilibrium point asymptotically in time, ie. for $t \to \infty$ or $t \to -\infty$. Cannot reach an equilibrium point at finite time

A equilibrium point is a trajectory

Trajectories cannot cross

It follows that the equilibrium point can be only reached for $t \to \infty$ or $t \to -\infty$



Finding equilibria

Example Find all equilibria of the following dynamical system

$$\dot{y}_1 = 4y_1y_2 - 2$$

$$\dot{y}_2 = (y_1 - 2)(y_2 - 2y_1)$$

Solution At an equilibrium point $(y_1, y_2) = (y_1^*, y_2^*)$ we must have $\dot{y}_1 = 0$ and $\dot{y}_2 = 0$

$$\begin{cases} \dot{y}_1 = 4y_1y_2 - 2 = 0 \\ \dot{y}_2 = (y_1 - 2)(y_2 - 2y_1) = 0 \end{cases}$$

From $4y_1y_2 - 2 = 0 \Rightarrow y_1y_2 = \frac{1}{2}$

$(y_1 - 2)(y_2 - 2y_1) = 0 \Rightarrow$ Either $y_1 = 2$

or $y_2 = 2y_1$

- If $y_1 = 2$ $\quad y_1y_2 = \frac{1}{2} \Rightarrow y_2 = \frac{1}{4} \Rightarrow (y_1^*, y_2^*) = (2, \frac{1}{4})$

  1st equilibrium point

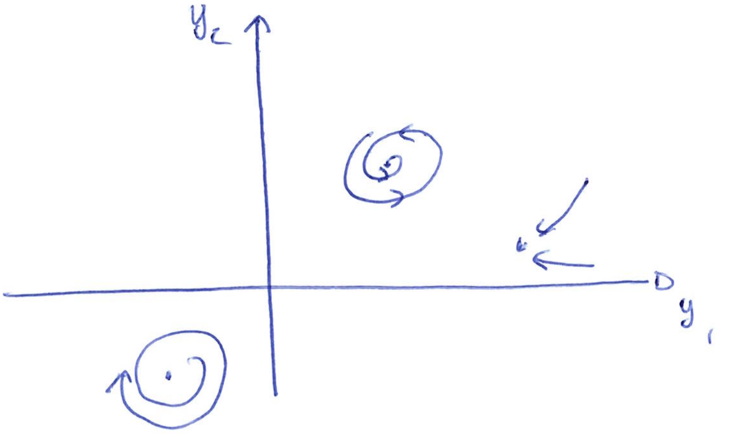
- If $y_2 = 2y_1$, $\quad y_1y_2 = \frac{1}{2} \Rightarrow 2y_1^2 = \frac{1}{2} \Rightarrow y_1^2 = \frac{1}{4}$

$\Rightarrow y_1 = \pm\frac{1}{2} \Rightarrow (y_1^*, y_2^*) = (\cancel{1}, \frac{1}{2}, 1)$ 2nd equilibrium point

$(y_1^*, y_2^*) = (\cancel{-1}, -\frac{1}{2}, -1)$ 3rd equilibrium point

Equilibria of a dynamical system

A dynamical autonomous system can have many equilibria.



3 equilibria, for example

Which are the typical trajectories close to these equilibria?

We can study these trajectories by linearisation (weeks 8-11)

In this module we will consider only ISOLATED equilibria,

for each equilibrium point (y_1^*, y_2^*) there exists $R>0$ such

that inside the circle $(y_1 - y_1^*)^2 + (y_2 - y_2^*)^2 = R^2$ there are no

other equilibria.

Linearization of autonomous dynamical systems (1st-order ODEs)

A linear dynamical system of 1st-order ODEs with two dependent variables y_1, y_2 reads

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} f_1(y_1, y_2) \\ f_2(y_1, y_2) \end{pmatrix} = \begin{pmatrix} a_{11} y_1 + a_{12} y_2 \\ a_{21} y_1 + a_{22} y_1 \end{pmatrix}$$

or equivalently

LINEAR DYNAMICAL SYSTEMS

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

where A is a 2×2 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

with $a_{11}, a_{12}, a_{21}, a_{22} \in \mathbb{R}$

A autonomous dynamical system that is NOT linear is called NON-LINEAR.

Examples

$$\begin{cases} \dot{y}_1 = 4y_1 - 2y_2 \\ \dot{y}_2 = 3y_1 + y_2 \end{cases}$$

Linear!

$$\begin{cases} \dot{y}_1 = 4y_1 y_2 - 2 \\ \dot{y}_2 = (y_1 - 2)(y_2 - 2y_1) \end{cases}$$

Non-linear!