Main Examination period 2020 – January – Semester A

MTH5123: Differential Equations

Duration: 2 hours

Student number | | | | | | | | | Desk number | | |

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Write your solutions in the spaces provided in this exam paper. If you need more paper, ask an invigilator for an additional booklet and attach it to this paper at the end of the exam.

You should attempt ALL questions. Marks available are shown next to the questions.

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Complete all rough work in the answer book and cross through any work that is not to be assessed.

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Exam papers must not be removed from the examination room.

Examiners: Weini Huang, Shabnam Beheshti

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Do not write on it.

Question	Mark	Comments
1	/ 15	
2	/ 20	
3	/ 25	
4	/ 15	
5	/ 25	
Total		

Question 1 [15 marks].

Let r be the per capita growth rate of a population in the time interval dt and N be the population density, which is the total number of individuals in this population. Note r is a constant number not a variable here.

(a) Find the general solution to the first-order ordinary differential equation (ODE), $\frac{dN}{dt} = rN$. Supposing $r = 1$, find the solution of this ODE when the initial population density ($t = 0$) satisfies $N(0) = 100$. **[5]**

(b) Suppose the per capita growth rate will decrease linearly with the population density. When the population density approaches its maximum size K, the per capita growth rate decreases to 0. This yields the logistic equation,

$$\frac{dN}{dt} = N(1 - \frac{N}{K}).$$

Find the general solution of this ODE by the method of the separation of variables. Note, K is a constant number not a variable here. According to this general solution, describe how the population size changes as $t \to \infty$. **[10]**

Write your solutions here

Continue-1: solutions to question 1

Write your solutions here

Continue-2: solutions to question 1

Write your solutions here

Question 2 [20 marks].

(a) Find the general solution of the following ordinary differential equation

$$(x-1)y' = 2y.$$

[5]

(b) Use the Picard-Lindelöf Theorem to show that $0 < A < 1$ is required for the ODE in (a) with the initial condition $y(0) = 1$ to have a unique solution in a rectangular space $D = \{|x| \leq A, |y-1| \leq B\}$. Sketch the position of this rectangle D in the xy plane. Find out all other conditions between A and B to guarantee the uniqueness of the solution in D. **[8]**

(c) Use the Picard-Lindelöf Theorem to show whether there exists a unique solution to the ODE in (a) with a different initial condition $y(1) = 0$. If not, based on the general solution obtained in (a) and this initial condition $y(1) = 0$, sketch and describe all possible solutions to this initial value problem in the xy plane. **[7]**

Write your solutions here

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Continue-1: solutions to question 2

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Continue-2: solutions to question 2

Write your solutions here

Continue-3: solutions to question 2

Write your solutions here

Question 3 [25 marks].

(a) Find the general solution to the homogeneous second-order linear ODE

$$2y'' + y' - 15y = 0.$$

[5]

(b) Use the solution in (a) to find the general solution to the inhomogeneous second-order linear ODE

$$2y'' + y' - 15y = 6e^{-2x}.$$

[10]

(c) Use the variation of parameter method to find the general solution of the inhomogeneous equation

$$y'' - 5y' + 6y = e^{3x} \cos x.$$

[10]

Write your solutions here

Continue-1: solutions to question 3

Write your solutions here

Continue-2: solutions to question 3

Write your solutions here

Continue-3: solutions to question 3

Write your solutions here

Question 4 [15 marks].

(a) Write down the general solution to the following Euler-type second order differential equation

$$x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = 0.$$

[9]

(b) Find the solution to the following Boundary Value Problem,

$$y'' + 9y = 0,\ y'(0) = 5,\ y(\frac{\pi}{3}) = -\frac{5}{3}.$$

[6]

Write your solutions here

Continue-1: solutions to question 4

Write your solutions here

Continue-2: solutions to question 4

Write your solutions here

Question 5 [25 marks].

Consider a system of two nonlinear first-order ODEs,

$$\dot{x} = xy - 4, \quad \dot{y} = (x-4)(y-x).$$

(a) Compute all equilibria of this ODE system. **[6]**

(b) Linearise the above equations around the equilibrium at $y = 2$ and write down the resulting linear system in matrix form. Find the corresponding eigenvalues and eigenvectors to this linearised system and write down its general solution. **[10]**

(c) Determine the type of equilibrium of the original ODE system at $y = 2$ (center, stable node sink, stable spiral, saddle, unstable node source, unstable spiral) and sketch the phase portrait for the linearised system in (b). **[4]**

(d) Using the result in (b), find the solution of this linearised system in (b) corresponding to the initial conditions $x(0) = 3 + \sqrt{2}$, $y(0) = 3$. Determine the tangent vector to the trajectory of the solution at $t = 0$ and the value of x as $t \to \infty$ for this specific initial condition. **[5]**

Write your solutions here

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Continue-1: solutions to question 5

Write your solutions here

Continue-2: solutions to question 5

Write your solutions here

End of Paper – An appendix of 1 page follows.

Formula Sheet

Partial fraction decomposition

$$\frac{A}{(A-B)B} = \frac{1}{A-B} + \frac{1}{B}$$

Integrations

$$\int e^x \cos x dx = \frac{1}{2} e^x (\sin x + \cos x)$$

Picard-Lindelöf Theorem. Let $\mathcal{D}$ be the rectangular space in the xy plane defined as $\mathcal{D} = (|x-a| \leqslant A,\ |y-b| \leqslant B)$ and suppose $f(x,y)$ is a function defined on $\mathcal{D}$ which satisfies the following conditions:

(i) $f(x,y)$ is continuous and therefore bounded in $\mathcal{D}$. The parameters A and B satisfy $A \leq B/M$ where $M = max_{\mathcal{D}}|f(x,y)|$

(ii) $|\frac{\partial f}{\partial y}|$ is bounded in $\mathcal{D}$.

Then there exists a unique solution on $\mathcal{D}$ to the initial value problem

$$\frac{dy}{dx} = f(x,y), \quad y(a) = b.$$

End of Appendix.