Variation of parameter method:

Given a 2^{nd}-order linear inhomogeneous ODE of the type

$$a_2 y'' + a_1 y' + a_0 y = f(x)$$

with <u>two distinct roots</u> $\lambda_1 \neq \lambda_2$ of the characteristic equation

$$M_2(\lambda) = a_2 \lambda^2 + a_1 \lambda + a_0 = 0$$

The general solution $y_g(x)$ can be written as

$$y_g(x) = y_p(x) + y_h(x)$$

where $y_h(x)$ is the general solution of the homogeneous ODE

$$a_2 y'' + a_1 y' + a_0 y = 0$$

and $y_p(x)$ is a particular solution of the inhomogeneous ODE.

With the variation of parameter method we found one particular solution $y_p(x)$

$$y_p(x) = \frac{1}{a_2(\lambda_1 - \lambda_2)} \left\{ e^{\lambda_1 x} \underbrace{\int f(x) e^{-\lambda_1 x} dx}_{F_1(x)} - e^{\lambda_2 x} \underbrace{\int f(x) e^{-\lambda_2 x} dx}_{F_2(x)} \right\}$$

while $y_h(x)$ is given by

$$y_h(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$ with c_1, c_2 arbitrary constants

Solve the ODE

$$\cancel{y''} \quad y''-3y'+2y=e^{2x} \qquad\qquad f(x)=e^{2x}$$

Consider the characteristic equation of the corresponding homogeneous ODE $\quad y''-3y'+2y=0$

$$M_2(\lambda)=\lambda^2-3\lambda+2=0$$

$\Rightarrow$ Roots $\quad \lambda=\dfrac{3\pm\sqrt{9-8}}{2}=\dfrac{3\pm1}{2} \left\langle \begin{array}{l} \lambda_1=2 \\ \lambda_2=1 \end{array}\right.$

$\lambda_1\neq\lambda_2 \quad \Rightarrow$ Use the variation of parameter method.

The general solution will be given by

$$y_g(x)=y_p(x)+y_h(x)$$

where $\quad y_h(x)=c_1e^{2x}+c_2e^{x} \quad$ with $c_1,c_2\in\mathbb{R}$ arbitrary constants.

and $\quad y_p(x)=\dfrac{1}{1\cdot(2-1)}\left\{e^{2x}F_1(x)-e^{x}F_2(x)\right\}=e^{2x}F_1(x)-e^{x}F_2(x)$

$$F_1(x)=\int f(x)e^{-\lambda_1x}\,dx=\int e^{2x}e^{-2x}\,dx=\int dx=x$$

$$F_2(x)=\int f(x)e^{-\lambda_2x}\,dx=\int e^{2x}e^{-x}\,dx=\int e^{x}\,dx=e^{x}$$

$$y_p(x) = xe^{2x} - e^{x} \cdot e^{x} = xe^{2x} - e^{2x}$$

$$y_g(x) = xe^{2x} - e^{2x} + c_1 e^{2x} + c_2 e^{x}$$ with $c_1, c_2 \in \mathbb{R}$ arbitrary constant.

$$y_g(x) = xe^{2x} + \underbrace{(c_1 - 1)}_{C_1} e^{2x} + \underbrace{c_2}_{C_2} e^{x}$$

$$\boxed{y_g(x) = xe^{2x} + C_1 e^{2x} + C_2 e^{x}}$$

The educated guess method cannot be used because

$f(x) = e^{ax}$ where a coincides with one of the roots of the characteristic equation.