Corresponding homogeneous BVP.

Let us consider the BVP

(A) ODE:

$$a_2(x)\, y'' + a_1(x)\, y' + a_0(x)\, y = f(x)$$

$a_2(x) \neq 0$, $a_1(x)$, $a_0(x)$, $f(x)$ continuous for $x \in [x_1, x_2]$

(B) Linear BCs:

$\alpha, \beta, \gamma, \delta, b_1, b_2 \in \mathbb{R}$

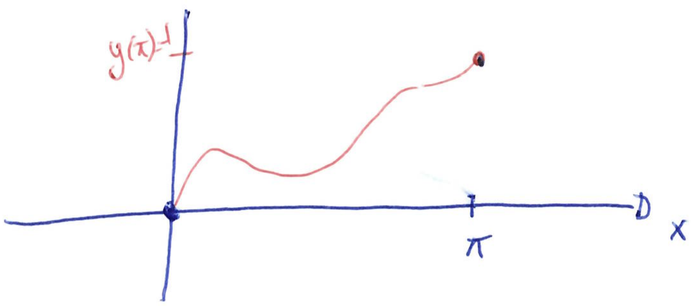
$$\alpha y'(x_1) + \beta y(x_1) = b_1 \qquad (\alpha, \beta) \neq (0,0)$$

$$\gamma y'(x_2) + \delta y(x_2) = b_2 \qquad (\gamma, \delta) \neq (0,0)$$

Given an inhomogeneous BVP the corresponding homogeneous BVP can be obtained by putting

$$\boxed{f(x) = 0 \qquad b_1 = b_2 = 0}$$

Example $\quad y'' + y = 0 \quad$ BC: $\begin{cases} y(0) = 0 \\ y(\pi) = 1 \end{cases}$



Solving this BVP means finding $y(x)$ that solves the ODE and the BVP

IF ANY

The corresponding homogeneous BVP

ODE: $y'' + y = 0$ BC: $\begin{cases} y(0) = 0 \\ y(\pi) = 0 \end{cases}$

The BVP can admit only

$$\left.\begin{array}{l} \text{(1) One solution} \\ \text{(2) No solution} \\ \text{(3) Infinite solutions} \end{array}\right\} \text{The Theorem of the Alternative will help us to predict which of these cases apply to a given BVP}$$

<u>Let us first solve some BVP!</u>

Example 1 Solve the BVP

ODE: $y'' + y = 0$ BCs: $\begin{cases} y(0) = 0 \\ y(\pi) = 1 \end{cases}$

(1) We first find the general solution to the ODE $y'' + y = 0$

$\Rightarrow$ Characteristic equation

$M_2(\lambda) = \lambda^2 + 1 = 0$ $\Rightarrow$ Roots $\lambda^2 = -1$ $\quad \lambda_1 = +i$, $\lambda_2 = -i$

$\lambda_1 = \alpha + i\beta$
$\lambda_2 = \alpha - i\beta$ $\quad \alpha = 0 \quad \beta = 1$

General solution $y_g(x) = e^{\alpha x}(B \sin \beta x + A \cos \beta x)$

$$\boxed{y_g(x) = A \cos x + B \sin x}$$

with $A, B \in \mathbb{R}$ arbitrary constant

(2) Impose the BCs $\begin{cases} y(0) = 0 \\ y(\pi) = 1 \end{cases}$

$0 = y(0) = A \underbrace{\cos 0}_{1} + B \underbrace{\sin 0}_{0} = A \Rightarrow \boxed{A = 0}$

$1 = y(\pi) = A \underbrace{\cos \pi}_{-1} + B \underbrace{\sin \pi}_{0} = -A \Rightarrow \boxed{A = -1}$

There is NO SOLUTION to the BVP because the BCs cannot be simultaneously satisfied for any choice of A and B.

Example 2 Solve the BVP

ODE: $y'' + y = 0$ BC: $\begin{cases} y(0) = 1 \\ y(\pi) = -1 \end{cases}$

① The general solution to the ODE $y'' + y = 0$ is

$$\boxed{y_g(x) = A\cos x + B\sin x}$$

where $A, B \in \mathbb{R}$ are arbitrary constant (see previous example)

② Impose the BCs $\begin{cases} y(0) = 1 \\ y(\pi) = -1 \end{cases}$

$$1 = y(0) = A\cos 0 + B\sin 0 = A \Rightarrow \boxed{A = 1}$$

$$-1 = y(\pi) = A\underbrace{\cos\pi}_{-1} + B\underbrace{\sin\pi}_{0} = -A \Rightarrow \boxed{A = 1}$$

The BVP has INFINITE solutions

$$\boxed{y(x) = \cos x + B\sin x}$$

where $B \in \mathbb{R}$ is an arbitrary constant

Example 3 Solve the homogeneous BVP corresponding to the BVP in examples 1 & 2.

ODE: $y'' + y = 0$ BC: $\begin{cases} y(0) = 0 \\ y(\pi) = 0 \end{cases}$

① The general solution to $y'' + y = 0$

$$y_g(x) = A\cos x + B\sin x$$ with $A, B \in \mathbb{R}$ arbitrary constants.

② We impose the BCs.

$$0 = y(0) = A\cos 0 + B\sin 0 = A \Rightarrow \boxed{A = 0}$$

$$0 = y(\pi) = A\cos\pi + B\sin\pi = -A \Rightarrow \boxed{A = 0}$$

To satisfy the BCs we can impose $A = 0$ but B is arbitrary.

The BVP has INFINITE solutions

$$\boxed{y(x) = B\sin x}$$ with $B \in \mathbb{R}$ arbitrary constant

Finding

Example 1 and Example 2 are two examples of BVP with no solution and infinite number of solutions respectively.

Their corresponding homogeneous BVP (which is the same) has INFINITE solutions.

Example 4

ODE: $y'' + y = 0$ $\qquad \begin{cases} y(0) = 1 \\ y\left(\frac{\pi}{2}\right) = 1 \end{cases}$

1. Find the general solution to the ODE $y'' + y = 0$

$$y_g(x) = A\cos x + B\sin x$$ where $A, B \in \mathbb{R}$ are arbitrary constants

2. Impose the BCs:

$$1 = y(0) = A\cos 0 + B\sin 0 = A \implies \boxed{A = 1}$$

$$1 = y\left(\tfrac{\pi}{2}\right) = A\underbrace{\cos\tfrac{\pi}{2}}_{=0} + B\underbrace{\sin\tfrac{\pi}{2}}_{=1} = B \implies \boxed{B = 1}$$

The BVP has a UNIQUE solution

$$\boxed{y(x) = \cos x + \sin x}$$

Example 5

Let us consider the homogeneous BVP corresponding to the BVP of example 4.

ODE: $y'' + y = 0$ $\qquad$ BC: $\begin{cases} y(0) = 0 \\ y\left(\frac{\pi}{2}\right) = 0 \end{cases}$

1. The general solution to the ODE

$$y_g(x) = A\cos x + B\sin x$$ with $A, B \in \mathbb{R}$ and arbitrary constants.

② Let us impose the BCs:

$$0 = y(0) = A\cos 0 + B\sin 0 = A \implies \boxed{A=0}$$

$$0 = y\left(\frac{\pi}{2}\right) = A\underbrace{\cos\frac{\pi}{2}}_{=0} + B\underbrace{\sin\frac{\pi}{2}}_{=1} = B \implies \boxed{B=0}$$

The BVP has a UNIQUE (trivial) solution

$$\boxed{y=0}$$

<u>Finding</u>: Example 4 is an example of a BVP with a UNIQUE solution whose corresponding homogeneous BVP (Example 5) has a UNIQUE (trivial) solution.

How general are these examples?