Assessment

1^{st} Coursework — Opens . Thursday 3 Nov 5pm

10% — Deadline Thursday 10 Nov 5pm

Quiz based

Train - Mock quizzes.

2^{nd} Coursework — Either in week 10 or 11.

10%

Final exam — Invigilated, I.T. lab. No calculator.

80% — Train with final exams year 2020/2021 and 2021/2022.

Euler type ODEs

These are linear 2^{nd}-order homogeneous ODEs of the type

$$\boxed{a x^2 y'' + b x y' + c y = 0} \qquad (1)$$

$a, b, c \in \mathbb{R}$

These are NOT ODEs with constant coefficients but they are REDUCIBLE to ODE with constant coefficients.

This can be achieved by setting

$$x = e^t \quad \text{or} \quad t = \ln x \qquad \text{with } x > 0$$

and considering the function

$$z = z(t) = y(x(t))$$

Proposition: The function $z = z(t)$ defined as $z = y(x(t))$ with $x = e^t$ obeys the 2^{nd}-order linear homogeneous ODE with constant coefficient

$$\boxed{a \ddot{z} + (b-a)\dot{z} + c z = 0} \qquad (2)$$

Proof Let us calculat $\dot{z}$ and $\ddot{z}$

We have

$$\dot{z} = \frac{d}{dt} z(t) = \frac{d}{dt} y(x(t)) \overset{\text{chain rule}}{=} \left(\frac{dy(x)}{dx}\right) \frac{dx(t)}{dt} \overset{x=e^t}{=} y' \frac{de^t}{dt} = y' e^t$$

$$\boxed{\dot{z} = y' e^t}$$

$$\ddot{z} = \frac{d\dot{z}}{dt} = \frac{d}{dt}(y'e^t) \overset{\text{productrule}}{=} \left(\frac{d}{dt}y'\right) e^t + y' \underbrace{\left(\frac{de^t}{dt}\right)}_{=e^t}$$

$$\ddot{z} = \left(\frac{dy'}{dt}\right) e^t + y' e^t \quad (*)$$

Now using the chain rule we get

$$\frac{dy'}{dt} = \frac{d}{dt} y'(x(t)) = \frac{d}{dx} y'(x) \cdot \frac{dx}{dt} \overset{x=e^t}{=} y'' \frac{de^t}{dt} = y'' e^t \quad (**)$$

Therefore inserting $(**)$ into $(*)$ we get

$$\ddot{z} = \underbrace{(y'' e^t)}_{=\frac{dy'}{dt}} e^t + \underbrace{y' e^t}_{\dot{z}} \Rightarrow \ddot{z} = y'' e^{2t} + \dot{z}$$

$$\Rightarrow \boxed{\ddot{z} - \dot{z} = y'' e^{2t}}$$

Therefore we have

$$\begin{cases} \dot z = y' e^{t} \\ \ddot z - \dot z = y'' e^{2t} \end{cases} \qquad x = e^{t} \quad \Rightarrow \begin{cases} \dot z = y' x \\ \ddot z - \dot z = y'' x^2 \end{cases}$$

$$\Rightarrow \begin{cases} y' = \dot z\, x^{-1} \\ y'' = (\ddot z - \dot z)\, x^{-2} \end{cases} \qquad (***)$$

Recall the Euler equation

$$a x^2 y'' + b x y' + c y = 0 \qquad (1)$$

Inserting $(***)$ we get

$$a \cancel{x^2} (\ddot z - \dot z) \cancel{x^{-2}} + b \cancel{x}\, \dot z \cancel{x^{-1}} + c z = 0$$

$$\boxed{a \ddot z + (b-a) \dot z + c z = 0} \qquad (2)$$

$\square$

Solving Euler ODEs

<u>Example</u> $\quad x^2 y'' - 4 x y' + 6 y = 0$

(1) Recognize it is a Euler ODE of the type

$$a x^2 y'' + b x y' + c y = 0$$

$a = 1 \qquad b = -4 \qquad c = 6 \quad$ ✓

(2) We define $z = z(t)$ given by $z = y(x(t))$ where $x > 0$

$x = e^t \Rightarrow t = \ln x$.

$z(t)$ satisfies

$$a\ddot{z} + (b-a)\dot{z} + cz = 0$$

$$\ddot{z} - 5\dot{z} + 6z = 0$$

(3) Solve $\ddot{z} - 5\dot{z} + 6z = 0$

Solve ODE: Calculate the characteristic equation

$$M_2(\lambda) = \lambda^2 - 5\lambda + 6 = 0$$

$$\lambda = \frac{5 \pm \sqrt{25-24}}{2} = \frac{5 \pm 1}{2} \begin{cases} \lambda_1 = 3 \\ \lambda_2 = 2 \end{cases}$$

(A) The general solution

$$z_g(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

$$z_g(t) = c_1 e^{3t} + c_2 e^{2t}$$ with $c_1, c_2 \in \mathbb{R}$ arbitrary constant.

(4) To find $y(x)$ we impose

$$y_g(x) = z_g(t)\Big|_{t = \ln x}$$ with $x > 0$

In our case

$$z_g(t) = c_1 e^{3t} + c_2 e^{2t}$$

$$y_g(x) = c_1 e^{3 \ln x} + c_2 e^{2 \ln x} = c_1 x^3 + c_2 x^2$$

$$\boxed{y_g(x) = c_1 x^3 + c_2 x^2}$$

with $c_1, c_2 \in \mathbb{R}$ arbitrary constants