General solution of the homogeneous linear 2^{nd}-order ODEs with constant coefficients.

We want to find the general solution to

$$a_2(x)\,y'' + a_1(x)\,y' + a_0(x)\,y = 0 \qquad \text{with } a_2 \neq 0 \quad a_1, a_0 \in \mathbb{R}$$

① Write the characteristic equation

$$M_2(\lambda) = a_2 \lambda^2 + a_1 \lambda + a_0 = 0$$

② Find the roots of $M_2(\lambda) = 0$

$$\lambda_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4 a_0 a_2}}{2 a_2}$$

③ (A) If $\lambda_1, \lambda_2 \in \mathbb{R}$ $\quad \lambda_1 \neq \lambda_2$

$$\boxed{y_g(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}}$$

with $c_1, c_2 \in \mathbb{R}$ arbitrary constants

(B) If $\lambda_1, \lambda_2 \in \mathbb{C}$ $\quad \lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$ $\quad$ with $\alpha, \beta \in \mathbb{R}$, $\beta \neq 0$

$$\boxed{y_g(x) = e^{\alpha x}\left(A \cos\beta x + B \sin \beta x\right)}$$

with $A, B \in \mathbb{R}$ arbitrary constants

or

$$\boxed{y_g(x) = c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x}}$$

with $c_1, c_2 \in \mathbb{C}$ arbitrary constants

(c) If $\lambda_1 \in \mathbb{R}$ is the $M_2(\lambda) = 0$ with multiplicity two

$$\boxed{y_g(x) = e^{\lambda_1 x}(c_1 x + c_2)} \qquad \text{with } c_1, c_2 \in \mathbb{R}$$

In case (c) you can show that

$$y_1 = x e^{\lambda_1 x} \quad , \quad y_2 = e^{\lambda_1 x}$$

$$a_2 y'' + a_1 y' + a_0 y = 0$$

Such that the charecteristic equation is

$$a_2 \lambda^2 + a_1 \lambda + a_0 = a_2 (\lambda - \lambda_1)^2 = 0$$

a_2

$a_1 = -2 a_2$

$a_0 = a_2$

Examples $\quad y'' - 4y' + 3y = 0 \qquad a_2 y'' + a_1 y' + a_0 y = 0$

$a_2 = 1, \quad a_1 = -4 \quad a_0 = 3$

1 Characteristic equation

$$\boxed{M_2(\lambda) = \lambda^2 - 4\lambda + 3 = 0}$$

(2) Find the roots $M_2(\lambda)=0$

$$\lambda = \frac{4 \pm \sqrt{16-12}}{2} = \frac{4 \pm \sqrt{4}}{2} = \begin{cases} \frac{4+2}{2} = 3 & \lambda_1 = 3 \\ \frac{4-2}{2} = 1 & \lambda_2 = 1 \end{cases}$$

(3) $\lambda_1, \lambda_2 \in \mathbb{R}$ $\lambda_1 \neq \lambda_2$

$$y_g(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \quad \Bigg| \quad \boxed{y_g(x) = c_1 e^{3x} + c_2 e^{x}}$$

$c_1, c_2 \in \mathbb{R}$

Example $y'' - 2y' + y = 0$

(1) Write characteristic equation

$$M_2(\lambda) = \lambda^2 - 2\lambda + 1 = 0 \qquad M_2(\lambda) = (\lambda-1)^2 = 0$$

(2) Find the roots

$$\lambda = \frac{2 \pm \sqrt{4-4}}{2} = 1 \qquad \boxed{\lambda_1 = 1}$$

(3) (c) $\lambda_1 \in \mathbb{R}$ with $M_2(\lambda_1) = 0$ with multiplicity two

$$y_g(x) = e^{\lambda_1 x}(c_1 x + c_2) \quad \Big| \quad \boxed{y_g(x) = e^{x}(c_1 x + c_2)}$$

c_1, c_2 real

Example $\quad \frac{1}{2}y'' + y' + y = 0$

① Write the characteristic equation

$$\boxed{M_2(\lambda) = \frac{1}{2}\lambda^2 + \lambda + 1 = 0}$$

② Find the roots

$$\lambda = \frac{-1 \pm \sqrt{1 - \cancel{4}^{2} \frac{1}{\cancel{2}} \cdot 1}}{\cancel{2} \cdot \frac{1}{\cancel{2}}} = -1 \pm \sqrt{-1} = \begin{cases} -1+i \\ -1-i \end{cases}$$

$\lambda_1 = -1 + i \quad \lambda_2 = -1 - i \qquad \alpha = -1 \qquad \beta = 1$

$\lambda_1 = \alpha + \beta i \quad \lambda_2 = \alpha - \beta i$

③ (B) The general solution is

$$y_g(x) = e^{\alpha x}(A \cos \beta x + B \sin \beta x)$$

$$\boxed{y_g(x) = e^{-x}(A \cos x + B \sin x)} \qquad A, B \in \mathbb{R}$$

or

$$y_g(x) = C_1 e^{(\alpha + \beta i)x} + C_2 e^{(\alpha - \beta i)x}$$

$$\boxed{y_g(x) = C_1 e^{(-1+i)x} + C_2 e^{(-1-i)x}} \qquad C_1, C_2 \in \mathbb{C}$$

The IVP of the homogeneas linear 2^{nd}-order ODEs with constant coefficient.

We consider IVP

ODE: $a_2 y'' + a_1 y' + a_0 = 0$ $\qquad a_2, a_1, a_0 \in \mathbb{R}$, $a_2 \neq 0$

IC: $\begin{cases} y(a) = b_1 \\ y'(a) = b_2 \end{cases}$

The IVP satisfies the hypothesis of the generalized Picard-Lindelöf theorem, ie. the IVP has a unique solution everywher, $x \in \mathbb{R}$

$\Rightarrow$ The I.C. determines uniquely the two arbitrary constants in the general solution

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Example Solve the IVP

$10y'' - y' - 3y = 0$ $\qquad$ I.C. $\begin{cases} y(0) = 1 \\ y'(0) = 0 \end{cases}$

1. Characteristic equation

$$M_2(\lambda) = 10\lambda^2 - \lambda - 3 = 0$$

Solve the IVP for

$$10y'' - y' - 3y = 0 \qquad \begin{cases} y(0) = 1 \\ y'(0) = 0 \end{cases}$$

(1) Characteristic equation

$$M_2(\lambda) = 10\lambda^2 - \lambda - 3 = 0$$

(2) Finds the roots

$$\lambda = \frac{1 \pm \sqrt{1+120}}{20} = \frac{1 \pm \sqrt{121}}{20} = \frac{1 \pm 11}{20} = \begin{cases} \frac{12}{20} = \frac{3}{5} \\ \frac{-10}{20} = -\frac{1}{2} \end{cases}$$

$$\lambda_1 = \frac{3}{5} \qquad \lambda_2 = -\frac{1}{2}$$

(3) General solution

$$y_g(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

$$\boxed{y_g(x) = c_1 e^{\frac{3}{5}x} + c_2 e^{-\frac{1}{2}x}}$$ with $c_1, c_2 \in \mathbb{R}$

(4) Impose the I.C. $\begin{cases} y(0) = 1 \\ y'(0) = 0 \end{cases}$

$$y(x) = c_1 e^{\frac{3}{5}x} + c_2 e^{-\frac{1}{2}x}$$

$$y'(x) = \frac{3}{5} c_1 e^{\frac{3}{5}x} - \frac{1}{2} c_2 e^{-\frac{1}{2}x}$$

$$\begin{cases} 1 = y(0) = c_1 + c_2 \\ 0 = y'(0) = \frac{3}{5} c_1 - \frac{1}{2} c_2 \end{cases}$$

$$\begin{cases} 1 = c_1 + c_2 \\ 0 = \frac{3}{5} c_1 - \frac{1}{2} c_2 \end{cases} \qquad \begin{cases} c_1 = 1 - c_2 \\ \frac{3}{5}(1 - c_2) - \frac{1}{2} c_2 = 0 \end{cases}$$

② Find the roots

$$\lambda = \frac{1 \pm \sqrt{1 + 4 \cdot 30}}{20} = \frac{1 \pm \sqrt{1+120}}{20} = \frac{1 \pm \sqrt{121}}{20} = \frac{1 \pm 11}{20}$$

$$\lambda_1 = \frac{1+11}{20} = \frac{12}{20} = \frac{3}{5}$$

$$\lambda_2 = \frac{1-11}{20} = \frac{-10}{20} = -\frac{1}{2}$$

③ General solution (B) $y_g(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$

$$\boxed{y_g(x) = C_1 e^{3/5 x} + C_2 e^{-1/2 x}}$$

$C_1, C_2 \in \mathbb{R}$

④ Imposing the I.C.

$$\begin{cases} y(0) = 1 \\ y'(0) = 0 \end{cases}$$

$$y(x) = C_1 e^{3/5 x} + C_2 e^{-1/2 x}$$

$$y'(x) = C_1 \frac{3}{5} e^{3/5 x} - C_2 \frac{1}{2} e^{-1/2 x}$$

$$1 = y(0) = C_1 e^0 + C_2 e^0 = C_1 + C_2$$

$$0 = y'(0) = C_1 \frac{3}{5} e^0 - C_2 \frac{1}{2} e^0 = \frac{3}{5} C_1 - \frac{1}{2} C_2$$

$$\begin{cases} 1 = C_1 + C_2 \\ 0 = \frac{3}{5} C_1 - \frac{1}{2} C_2 \end{cases}$$

$$\begin{cases} c_1 = 1 - c_2 \\ \frac{3}{5}(1 - c_2) - \frac{1}{2} c_2 = 0 \end{cases} \qquad \begin{cases} c_1 = 1 - c_2 \\ \frac{3}{5} - \left(\frac{1}{2} + \frac{3}{5}\right) c_2 = 0 \end{cases},$$

$$\begin{cases} c_2 = 6/11 \\ c_1 = 1 - c_2 = 5/11 \end{cases} \qquad \begin{matrix} c_1 = 5/11 \\ c_2 = 6/11 \end{matrix}$$

The solution to the IVP

$$\boxed{y(x) = \frac{5}{11} e^{3/5 x} + \frac{6}{11} e^{-1/2 x}}$$