Differential Equations
Lesson 1 Week 1

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Code of conduct

All lectures are in presence but they are also streamed on Zoom

Also if you are in class you can login online to the Zoom link you will find in the Online Module Class in QM+

However whenever in class keep your computer/mobile/tablet

muted and switch off your audio at all times

Code of conduct

You can ask questions at any time

Questions are welcome!

However do not ask question before lectures

(as I need my full attention to work out the IT set up)

If you are following the lesson on Zoom and want to ask a question,

please use the “Raise Hand” bottom

Lectures and Tutorials

- Schedule:

- Lesson 1-2 Tuesday 11:00-13:00 Live Arts Two LT
- Lesson 3 Tuesday 14:00-15:00 Live Arts Two LT

You will be assigned to one of the following three tutorials
(check your timetable to know which applies)
Tutorials will start from week 1

- Tutorial 1: Thursday 12:00-13:00 PP2
- Tutorial 2: Thursday 16:00-17:00 Bancroft 1.13
- Tutorial 3: Thursday 17:00-18:00 Engineering 2.09

Lectures and Tutorials

Your tasks:

- Make an effort to attend the lecture live
- Take your own notes while attending the lecture
- Ask questions
- Complete formative assements, and mock quizzes and submit courseworks

Lecture notes

- Lecture note material available on QMPlus:
  - Typed in lecture notes
  - Handwritten lecture notes
- Your tasks:
  - Read the typed in lecture notes in advance of the lectures
  - Study the notes after the lecture to check your full understanding of the module material.

Reading list

- J. C. Robinson: An introduction to Ordinary Differential Equations (Cambridge University Press)
- Available from the library! (See link in QM+)

Formative Assessment

- Each week, at the tutorial we will cover
- The formative assignment and the mock quiz for the week, available on the QMPlus page
- Your task:
- Attempt the formative assessment and the mock quiz before the tutorial
- Ask questions regarding the coursework during the tutorial

Assessed courseworks

- Every assessed coursework is worth 10% of the final marks
- You will have one week to complete assessed courseworks
- The 2 assessed courseworks will be posted in weeks 6,11

Final exam

- The final exam will account for 80% of the final mark

Feedback

Feedback on your assessed coursework:

Personalised feedback

Quiz questions: You will be able to see your scores soon after the submission deadline

General feedback

General feedback will be given during the online tutorial

where we will go over the most challenging questions of the assessed

courseworks and the common mistakes.

Online Forum

For any question on the module material that you would like to ask there are two ways to received feedback and answers:

- You can ask the question during the live lectures and tutorials
- You can post the question on the online forum

Participation to the online forum is highly beneficial as it increases the interactions between fellow students.

- The online forum will be monitored twice a week

Outline of the Lesson

- ODEs versus PDEs
- Dependent and independent variables
- ODEs and their order
- What is a solution of a ODE?
- The simplest linear ODE
- Examples of applications

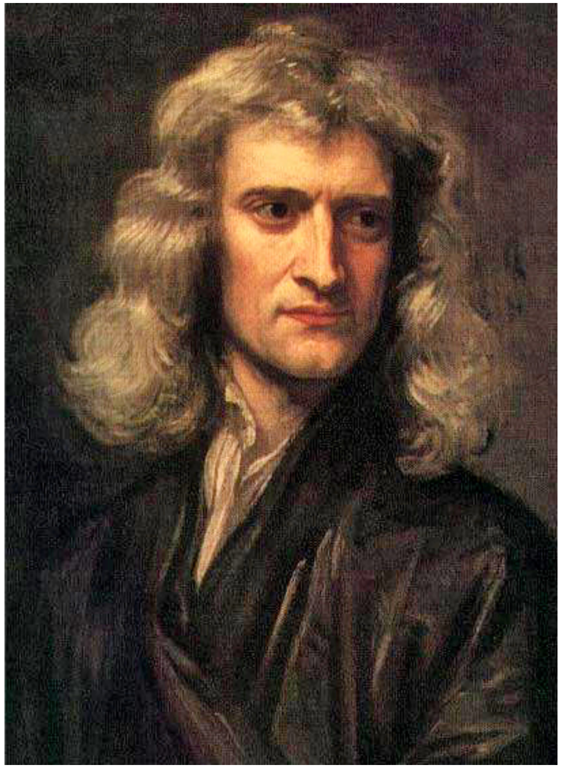
Isaac Newton
(25 December 1642 – 20 March 1726)

Driven by the need to understand

the physical word

including gravitation and mechanics

Isaac Newton founded Calculus



Newton groundbreaking role in mathematics and physics

Newton groundbreaking role in mathematics and physics can be summarised by the following two points:

- He wrote the differential equations describing the motion of masses in presence of forces (such as gravitation) valid in classical mechanics

- He founded Calculus for solving the differential equations that he formulated

In this module we will expand your analytical skills to solve differential equations and to solve problems coming from several applications

Ordinary differential equations (ODEs)

Ordinary differential equations are equations

satisfied by a function of a single variable

involving the function, its derivatives and its argument

Example: $\dfrac{dy}{dx} = x^2$

This equation is solved when we find all the functions $y(x)$ that satisfy it

x indicates the independent variable

y indicates the dependent variable.

Partial differential equations (PDEs)

Ordinary differential equations differs from partial differential equations

Partial differential equations are satisfied by functions of two or more variables and involve the function together with its partial derivatives and its argument

Example: $\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0$ (wave equation)

This equation is solved when we find all the functions $u(x, t)$ that satisfy it

x, t indicate the two independent variables

u indicates the dependent variable.

In this module we will cover only ODEs

The notation

The independent variable and the dependent variable

can be also indicated by other letters:

Function	Independent variable	Dependent variable
$y(x)$	x	y
$y(t)$	t	y
$x(t)$	t	x

Derivative notation

To indicate the derivative we can use the following notation

Derivative with respect to x

$$\frac{dy}{dx} \longleftrightarrow y'(x)$$

$$\frac{d^2y}{dx^2} \longleftrightarrow y''(x)$$

$$\vdots \qquad \vdots$$

$$\frac{d^ny}{dx^n} \longleftrightarrow y^{(n)}(x)$$

Derivative with respect to t

$$\frac{dy}{dt} \longleftrightarrow \dot{y}(t)$$

$$\frac{d^2y}{dt^2} \longleftrightarrow \ddot{y}(t)$$

$$\vdots \qquad \vdots$$

$$\frac{d^ny}{dt^n} \longleftrightarrow y^{(n)}(t)$$

Ordinary differential equations

An ordinary differential equation of order n for a function $y(x)$

is an equation of the form

$$\mathscr{F}\left(y^{(n)}(x), y^{(n-1)}(x), \ldots, y''(x), y'(x), y, x\right) = 0$$

(where the $n-th$ derivative $y^{(n)}(x)$ occurs in $\mathscr{F}$)

Order of a ODE

The order of an ODE

is the order to the highest derivative present in the equation

Examples:

$y' \sin x - y^2 + x^3 = 0$	$\mathscr{F}(y', y, x) = 0$	**1st order ODE**
$(y')^2 - y''' + e^x = 0$	$\mathscr{F}(y''', y', x) = 0$	**3rd-order ODE**
$e^x y'' + \sin y = 0$	$\mathscr{F}(y'', y, x) = 0$	**2nd-order ODE**

Normal form of an ODE

The normal form of an n-order ODE is an explicit expression of the n-order derivative of the type

$$y^{(n)}(x) = f(y^{(n-1)}, y^{(n-2)} \ldots, y'', y', y, x)$$

- **Examples**: 1st-order ODE in normal form $$y' = f(x, y)$$

  2nd-order ODE in normal form $$y'' = f(x, y, y')$$

Solution of an ODE

A solution of of a given ODE is a function for which the ODE becomes and identity

Example: The function $y = \bar{y}(x)$ is a solution of the ODE

$$\frac{dy}{dx} = f(x, y)$$

if

$$\frac{d\bar{y}(x)}{dx} = f(x, \bar{y}(x))$$

is an identity

You can use other notation for

the independent variable and the dependent variables!

Differential equations and variables

ODE	Independent variable	Dependent Variable	Normal form?	Order
$\frac{dy}{dt} = yt$	t	y	**Yes**	**1**
$\ddot{y} = -g$	t	y	**Yes**	**2**
$\ddot{x} = -\frac{k}{m}(x - x_0)$	t	x	**Yes**	**2**
$e^{-x}y' = y^2$	x	y	**No**	**1**

The simplest linear ODE

The simplest (linear) ODE you already know how to solve from Calculus is

$$\boxed{\frac{dy}{dx} = f(x)}$$

which has as solution the anti-derivative of $f(x)$

$$\boxed{y(x) = F(x) = \int f(x)dx + C}$$

Indeed we have the identity

$$\boxed{\frac{dF(x)}{dx} = f(x)}$$

Please refresh your Calculus I and Algebra

by

answering the

Revision questions of pre-request knowledge from previous modules

posted in the QM+ page under Week 1

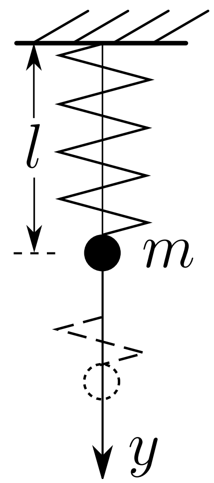
Applications

Physics

Newton Law

To solve problem in physics (classical mechanics) Newton formulated the three Newton Laws

For a mass hanging from the ceiling with a spring the Newton 2nd Law



is a ODE that reads

$$\boxed{m\ddot{y} = f(\dot{y}, y, t)}$$

where $f(\dot{y}, y, t)$ expresses all the forces acting on the mass:

the gravitational force, the elastic force (due to the spring) and friction

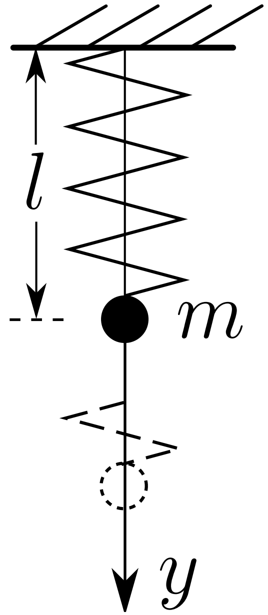
Example of forces

The Newton 2nd Law

$$m\ddot{y} = f(\dot{y}, y, t)$$

reads in this case

$$m\ddot{y} = mg - k(y - l) - \gamma\dot{y}$$



$$f_g = mg$$ **Gravitational force**

$$f_e = -k(y - l)$$ **Elastic force**

$$f_f = -\gamma\dot{y}$$ **Friction**

If the spring is not there

If the spring is not present the 2nd Law of Newton

$$m\ddot{y} = f(\dot{y}, y, t)$$

reads

$$m\ddot{y} = mg - \gamma\dot{y}$$

By putting

$$\dot{y} = v$$

The equation reduces to a 1st order separable equation

$$\boxed{m\dot{v} = mg - \gamma v}$$

Applications

Biology

Exponential growth model

The number N of individuals of a population grows in time t

by a constant pro-capita growth rate γ

$$\boxed{\frac{dN}{dt} = \gamma N}$$

with initial condition $N(0) = N_0$

In this model the population grows exponentially

$$\boxed{N(t) = N_0 e^{\gamma t}}$$

(see material of tutorial week 1 for explanation on the solution)

Logistic growth model

In realistic situations however the population of a species cannot growth indefinitely and exponentially because usually it has access only to finite resources.

In presence of finite resources the pro-capita growth rate $\tilde{\gamma}$

diminishes as the population approaches the carrying capacity K, i.e.

$$\boxed{\tilde{\gamma} = \tilde{\gamma}(N, t) = \gamma \left(1 - \frac{N}{K} \right)}$$

The logistic growth model

Therefore in presence of finite resources the population $N(t)$ follows the differential equation

$$\boxed{\frac{dN}{dt} = \gamma N \left(1 - \frac{N}{K}\right)}$$

Note that for $N = K$ **we have** $\dfrac{dN}{dt} = 0$

Solution (by separation of variables)

The ODE describing logistic growth of a population

$$\boxed{\frac{dN}{dt} = \gamma N \left(1 - \frac{N}{K}\right)}$$

with initial condition $N(0) = N_0$

can be solved by separation of variable (try as an exercise) getting

$$\boxed{N(t) = K \frac{N_0 e^{\gamma t}}{K - N_0 + N_0 e^{\gamma t}}}$$

Asymptotically in time the population saturates and reaches the carrying capacity

$$\boxed{\lim_{t \to \infty} N(t) = K}$$

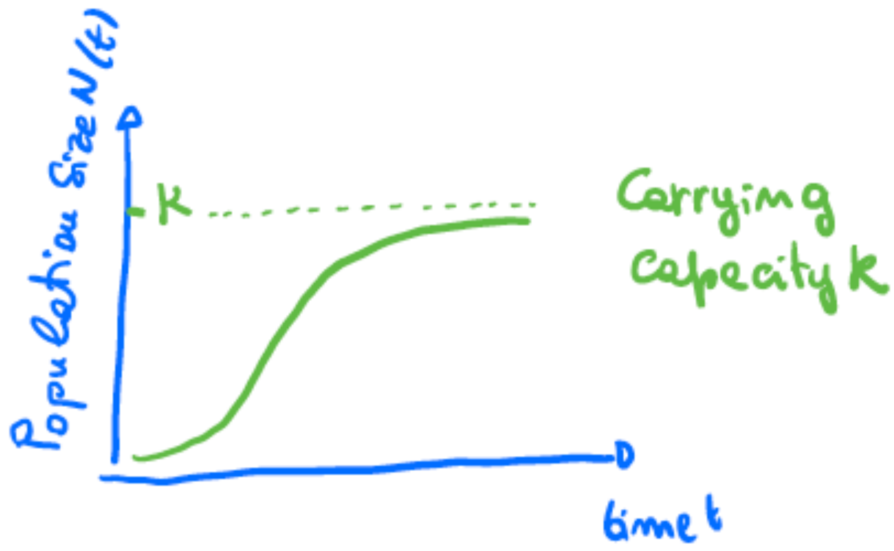
Exponential versus logistic growth model

Exponential growth

Constant growth rate

Logistic growth

Population-dependent growth rate



Differential equations are useful to solve

many applied problems!

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