MTH5123 Differential Equations

Solution to the Exam Problems 2016

All problems are either already seen in example classes, or moderate modifications of those.

1. **(a)** **(i)** Find all functions $f(y)$ for which the following differential equation becomes exact:

$$e^x f(y) + x^2 + (e^x \cos y + y) \frac{dy}{dx} = 0 \quad (1)$$

(4 points)

Solution: Denoting $P(x,y) = e^x f(y) + x^2, \quad Q(x,y) = e^x \cos y + y$ [**1p**] we can rewrite the above equation in the standard form $P(x,y) + Q(x,y)\frac{dy}{dx} = 0$. We then have $\frac{\partial P}{\partial y} = e^x f'(y)$ whereas $\frac{\partial Q}{\partial x} = e^x \cos y$ [**1p**], hence the equation is exact only if $f'(y) = \cos y$ or equivalently $f(y) = \sin y + C$ [**2p**], with any constant C.

(ii) Suppose, $f(y)$ is chosen so that the equation (1) is exact and $f(\pi) = 0$. Solve (1) in implicit form. (8 points)

Solution: The condition $f(\pi) = C = 0$, so that $f(y) = \sin y$ [**1p**]. Then the general solution should be looked for in implicit form as $F(x,y) = Const$ where

$$F = \int P(x,y)\,dx = \int \left(e^x \sin y + x^2\right) dx = e^x \sin y + \frac{x^3}{3} + g(y), \quad [\mathbf{2p}]$$

where $g(y)$ is to be determined from the condition $Q = \frac{\partial F}{\partial y} = e^x \cos y + g'(y)$ [**1p**]. We therefore conclude that $g'(y) = y$ [**1p**] so that $g(y) = \frac{y^2}{2}$ [**2p**]. Thus the solution in implicit form is $e^x \sin y + \frac{x^3}{3} + \frac{y^2}{2} = Const$ [**1p**].

b) Consider the initial value problem (IVP)

$$(x+2)\frac{dy}{dx} + (y+2)^{2/3} = 0, \quad y(0) = b\,. \tag{2}$$

where b is a real parameter and we assume $b \geq -2$.

(i) Find the value of the parameter b such that the corresponding IVP may have more than one solution and explain your choice. Confirm your choice by giving explicitly at least two different solutions of the IVP for such a value of the parameter. (8 points)

Solution: First, we rewrite the above ODE in the standard form: $\frac{dy}{dx} = f(x,y)$, with $f(x,y) = -\frac{(y+2)^{2/3}}{x+2}$ [**1p**]. The solution to IVP is unique if the function $f(x,y)$ is continuous in some domain in the xy plane centered at the point with coordinates $x=0, y=b$ [**1p**] and the modulus of its partial derivative $|\frac{\partial f}{\partial y}| = \frac{2}{3}\frac{1}{|x+2|}(y+2)^{-1/3}$ is bounded in the same domain [**1p**]. The second condition is certainly violated if $y+2=0$, implying that for $b=-2$ we may expect non-uniqueness. [**1p**]. Solving the differential equation by the separation of variables method we get the general solution:

$$\int \frac{dy}{(y+2)^{2/3}} = -\int dx\,\frac{1}{x+2} \quad \Rightarrow 3(y+2)^{1/3} = -\ln|x+2| + C\,[\mathbf{2p}]$$

The "dangerous" initial condition $y(0) = -2$ fixes $C = \ln 2$, so that a solution to IVP is $y = \frac{1}{27}(\ln 2 - \ln|x+2|)^3 - 2\,$[**1p**].
On the other hand the constant solution $y(x) = -2$ solves the same IVP [**1p**].

(ii) Use the Picard-Lindelöf theorem to verify that the existence and uniqueness of the solution for the IVP (2) with $b=0$ is guaranteed in the rectangular domain $\mathcal{D} := \{|x| < A,\ |y| < B\}$ with $A = 1/2$ and $B = 1$. (5 points)

Solution: According to the Picard-Lindelöf theorem to ensure uniqueness and existence of the solution the right-hand side $f(x,y) = -\frac{(y+2)^{2/3}}{x+2}$ has to be continuous in $\mathcal{D}$, which is indeed the case since the dangerous point $x=-2$ is not in $\mathcal{D}$ (as $|-2| > 1/2$)[**1p**]. Further, the modulus of its partial derivative $|\frac{\partial f}{\partial y}| = \frac{2}{3}\frac{1}{|x+2|}(y+2)^{-1/3}$ is bounded in the same domain as $y=-2$ does not belong to $\mathcal{D}$ (as $|-2| > 1$)[**1p**]. Finally, the parameters A and

B must satisfy the inequality $A < B/M$, where $M = \max_{(x,y)\in D}|f(x,y)|$, which for given $A = 1/2$, $B = 1$ is equivalent to $M < 2$[**1p**]. This is indeed ensured for our choice of A, B as we have

$$M = \max_{(x,y)\in D} \frac{(y+2)^{2/3}}{|x+2|} = \frac{max_{|y|<1}(y+2)^{2/3}}{min_{|x|<\frac{1}{2}}|x+2|} = \frac{3^{2/3}}{3/2} = \frac{2}{3^{1/3}} < 2, \quad [\mathbf{2p}]$$

since obviously $3^{1/3} > 1$.

2. Write down the solution to the following Boundary Value Problem (BVP) for the second order non-homogeneous differential equation

$$x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = f(x), \quad y(1) = 0,\ y'(3) = 0$$

by using the Green's function method along the following lines:

a) Using that the left-hand side of the ODE is in the form of an Euler-type equation determine the general solution of the associated homogeneous ODE. (6 points)

Solution: According to the general method of solving the Euler-type equation we introduce the new variable by $x = e^t$ and the new function $z(t)$ so that

$$z(t) = y(e^t), \quad \Rightarrow \quad \frac{dz}{dt} = e^t y', \qquad \frac{d^2z}{dt^2} = e^t y' + e^{2t} y''$$

From the above we find correspondingly that $y' = e^{-t}\dot{z}$, $y'' = e^{-2t}(\ddot{z} - \dot{z})$ [**1p**]. Substituting to the Euler-type equation reduces the latter to a homogeneous equation with constant coefficients: [**1p**]

$$e^{2t} \cdot e^{-2t}(\ddot{z} - \dot{z}) + 2e^t \cdot e^{-t}\dot{z} = \ddot{z} + \dot{z} = 0$$

The corresponding characteristic equation $\lambda^2 + \lambda = \lambda(\lambda + 1) = 0$ has two roots: $\lambda_1 = 0$ and $\lambda_2 = -1$ and the general solution is given by:

$$z(t) = C_1 + C_2 e^{-t}, \quad [\mathbf{1p}]$$

for arbitrary constants C_1 and C_2. Finally, substituting $t = \ln x$ gives [**1p**]

$$y(x) = C_1 + \frac{C_2}{x}, \quad \Rightarrow y'(x) = -\frac{C_2}{x^2}, \quad [\mathbf{2p}]\,.$$

b) Formulate the corresponding left-end and right-end initial value problems and use their solutions to construct the Green's function $G(x, s)$. (14 points)

Solution: The left-end boundary condition $y(1) = 0$ is imposed at $x_1 = 1$. By comparing it to the standard form $\alpha y'(x_1) + \beta y(x_1) = 0$ we conclude that $\alpha = 0, \beta = 1$ [**1p**]. Then the left-end initial value problem for the function $y_L(x)$ is formulated as

$$y_L(x_1) = \alpha, y_L'(x_1) = -\beta, \quad \Rightarrow \quad y_L(1) = 0, y_L'(1) = -1 \quad [\mathbf{2p}]$$

Substituting here the general solution of the homogeneous equation yields $C_2 = 1$, $C_1 = -1$ [**1p**] so that

$$y_L(x) = -1 + \frac{1}{x}, \qquad [\mathbf{1p}]$$

Obviously, $x_2 = 3$ and by comparing the right-end boundary condition $y'(3) = 0$ to the standard form $\gamma y'(x_2) + \delta y(x_2) = 0$ we conclude that $\gamma = 1, \delta = 0$[**1p**]. Then the right-end initial value problem for the function $y_R(x)$ is formulated as

$$y_R(x_2) = \gamma, y_R'(x_2) = -\delta, \quad \Rightarrow \quad y_R(3) = 1, y_R'(3) = 0, \quad [\mathbf{1p}]$$

which now gives $C_2 = 0$, $C_1 = 1$ and finally

$$y_R(x) = 1, \qquad [\mathbf{2p}]$$

Now we can use $y_L(x), y_R(x)$ for constructing the Green's function $G(x, s)$. First we calculate the Wronskian

$$W(s) = y_L(s)y_R'(s) - y_R(s)y_L'(s) = \frac{1}{s^2}, \quad [\mathbf{1p}]$$

We also should take into account that from the original ODE $a_2(s) = s^2$ so that $a_2(s)W(s) = 1$ [**1p**] and we have

$$A(s) = y_R(s)/\left(a_2(s)W(s)\right) = 1, \quad B(s) = y_L(s)/\left(a_2(s)W(s)\right) = -1+\frac{1}{s}, \quad [\mathbf{1p}]$$

Finally the Green's function is constructed as

$$G(x, s) = \begin{cases} A(s)y_l(x), & 1 \le x \le s \\ B(s)y_R(x), & s \le x \le 3 \end{cases}$$

$$= \begin{cases} -1 + \frac{1}{x}, & 1 \le x \le s \\ -1 + \frac{1}{s}, & s \le x \le 3 \end{cases}, \quad [\mathbf{2p}]$$

c) Write down the solution to the BVP in terms of $G(x, s)$ and $f(x)$ and use it to find the explicit form of the solution for $f(x) = x^2$.

(5 points)

Solution: The solution to the boundary value problem is given by

$$y(x) = \int_1^3 G(x,s)\, f(s)\, ds = \int_1^x G(x,s)\, f(s)\, ds + \int_x^3 G(x,s)\, f(s)\, ds$$

$$= \int_1^x \left(-1 + \frac{1}{s}\right) f(s)\, ds + \left(-1 + \frac{1}{x}\right) \int_x^3 f(s)\, ds\,, \quad [\mathbf{1p}]$$

substituting here $f(x) = x^2$ and using

$$\int_1^x \left(-1 + \frac{1}{s}\right) s^2\, ds = \left[-\frac{1}{3}s^3 + \frac{1}{2}s^2\right]_1^x = -\frac{1}{3}x^3 + \frac{1}{2}x^2 - \frac{1}{6}, \quad [\mathbf{1p}]$$

and

$$\int_x^3 s^2\, ds = \frac{1}{3}s^3|_x^3 = \frac{1}{3}(27 - x^3) = 9 - \frac{x^3}{3}, \quad [\mathbf{1p}]$$

we finally can write the solution in the form

$$y(x) = -\frac{1}{3}x^3 + \frac{1}{2}x^2 - \frac{1}{6} + \frac{1}{3}(27 - x^3)\left(-1 + \frac{1}{x}\right), \quad [\mathbf{1p}]$$

which after simplifying yields

$$y(x) = \frac{x^2}{6} - \frac{55}{6} + \frac{9}{x}, \quad [\mathbf{1p}]$$

3. Consider a system of two linear first-order ODE:

$$\dot{x} = -2x + y, \quad \dot{y} = -5x + 4y\,. \tag{3}$$

a) Determine eigenvalues and eigenvectors associated with the system, find equations for stable and unstable invariant manifolds and sketch the phase portrait. **(11 points)**

Solution. We rewrite the system in the matrix form $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$ where the matrix A associated with the system is given by $A = \begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix}$ [**1p**] The characteristic equation is $(-2-\lambda)(4-\lambda)+5 = \lambda^2 - 2\lambda - 3 = 0$ and has two real roots $\lambda_1 = 3$ and $\lambda_2 = -1$, [**1p**]. Eigenvector corresponding to $\lambda_1 = 3$ is found as

$$\begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix} \begin{pmatrix} p_1 \\ q_1 \end{pmatrix} = 3 \begin{pmatrix} p_1 \\ q_1 \end{pmatrix}, \quad [\mathbf{2p}]$$

which implies $-2p_1 + q_1 = 3p_1$, hence $q_1 = 5p_1$, [**1p**] and we can choose for example $p_1 = 1$ and $q_1 = 5$. For second eigenvalue $\lambda_2 = -1$ we similarly find

$$\begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix} \begin{pmatrix} p_2 \\ q_2 \end{pmatrix} = -1 \begin{pmatrix} p_2 \\ q_2 \end{pmatrix}, \quad [\mathbf{2p}]$$

which implies $q_2 = p_2$, [**1p**] so that we can choose, for example $p_2 = 1, q_2 = 1$. As $\lambda_1 > 0$ the trajectories will be for $t \to +\infty$ parallel to the straight line (the "unstable manifold") given by $y = \frac{q_1}{p_1}x = 5x$ [**1p**], whereas for $t \to -\infty$ they will be parallel to the stable manifold $y = \frac{q_2}{p_2}x = x$ [**1p**]. The corresponding phase portrait can be sketched as [**1p**]:

Sketch to be placed here

Description: a diagram of two intersecting invariant manifolds: $y = 5x$ (with the arrow showing motion along it away from the origin) and $y = x$ (with the arrow showing motion along it towards the origin)and separating the plane in 4 quadrants. The rest is a bunch of trajectories which are hyperbolas starting tangent to $y = x$ in all quadrants and flowing finally tangent to $y = 5x$.

b) For the nonlinear system

$$\dot{x} = f_1(x,y), \quad \dot{y} = f_2(x,y)$$

with

$$f_1(x,y) = (1-y)(2x-y), \quad f_2(x,y) = (2+x)(x-2y)$$

show that there exists an equilibrium point with $y = -4$ and determine its $x-$coordinate in the (x, y) plane. Linearize the system around such an equilibrium and determine its nature (stable vs. unstable) and type (saddle, focus, or node). Describe in words the shape of trajectories close to the point.

(9 points)

Solution. For $y = -4$ the right-hand sides $f_1(x,y), f_2(x,y)$ take the form $f_1 = 5(2x+4)$ and $f_2 = (2+x)(x+8)$. We see that for $x = -2$ both right-hand sides vanish simultaneously, hence this gives the coordinate of the equilibrium point in the (x,y) plane as $(-2,-4)$ [**1p**]. To linearize we need to evaluate $\frac{\partial f_1}{\partial x}, \frac{\partial f_1}{\partial y}, \frac{\partial f_2}{\partial x}, \frac{\partial f_2}{\partial y}$ at the point of equilibrium:

$$\frac{\partial f_1}{\partial x} = 2(1-y)|_{x=-2,y=-4} = 10, [\mathbf{1p}] \quad \frac{\partial f_1}{\partial y} = (-1-2x+2y)|_{x=-2,y=-4} = -5\,[\mathbf{1p}]$$

$$\frac{\partial f_2}{\partial x} = (2+2x-2y)|_{x=-2,y=-4} = 6\,[\mathbf{1p}]\,, \quad \frac{\partial f_2}{\partial y} = (-4-2x)|_{x=-2,y=-4} = 0\,[\mathbf{1p}]\,.$$

The linearized system in the matrix form $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$ where [**1p**] $A = \begin{pmatrix} 10 & -5 \\ 6 & 0 \end{pmatrix}$. The characteristic equation is now $(10-\lambda)(-\lambda) + 30 = \lambda^2 - 10\lambda + 30 = 0$ with the roots

$$\lambda_{1,2} = \frac{1}{2}\left(10 \pm \sqrt{100-120}\right) = 5 \pm i\sqrt{5}\,[\mathbf{1p}]\,.$$

We see that the eigenvalues are complex conjugate with positive real part, hence the equilibrium is an unstable focus [**1p**] and trajectories are spiralling away from the equilibrium point.[**1p**]

c) Consider a system of two nonlinear first-order ODE:

$$\dot{x} = -y + axy^2, \quad \dot{y} = x - bx^2 y\,. \tag{4}$$

where a, b are real constants. Find a relation between a and b such that the function $V(x,y) = \frac{1}{2}(x^2 + y^2)$ can be used as a Lyapunov function ensuring the stability of such a system in the whole (x, y) plane. (5 points)

Solution. The function $V(x,y) = \frac{1}{2}(x^2 + y^2) > 0$ for $(x,y) \neq (0,0)$ [**1p**] and its orbital derivative is given by

$$\mathcal{D}_f V = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y}$$

$$= x(-y+axy^2)+y\left(x - bx^2y\right) = -xy+ax^2y^2+xy-bx^2y^2 = (a-b)x^2y^2, \quad [\mathbf{3p}]$$

Therefore as long $a < b$ [**1p**] the orbital derivative $\mathcal{D}_f V \leq 0$ for all (x, y). For such values of parameters the function $V(x,y)$ is a valid Lyapunov function and ensures the stability of the solution of nonlinear equation in the whole (x, y) plane.

4. **a)** Find the general solution of the homogeneous ODE $y'' - 4y' + 13y = 0$ (4 points)

Solution: The characteristic equation is $\lambda^2 - 4\lambda + 13 = (\lambda - 2)^2 + 9 = 0$ [**1p**] which has two complex-conjugate roots: $\lambda_{1,2} = 2 \pm 3i$ [**1p**]. The general solution to the homogeneous equation is given by $y_h(x) = (c_1 \cos 3x + c_2 \sin 3x)e^{2x}$ with arbitrary constants c_1 and c_2. [**2p**].

b) Find the general solution of the non-homogeneous ODE

$$y'' - 4y' + 13y = 18e^{2x}$$

(8 points)

Solution: Since the functions e^{2x} is not a solution to the homogeneous equation[**1p**], we may use the "educated guess" method and look for the particular solution of the non-homogeneous equation in the form $y_p(x) = Ae^{2x}$[**2p**] so that:

$$y_p' = 2Ae^{2x}, \quad y_p''(x) = 4Ae^{2x}$$

Substituting this back to the nonhomogeneous equation gives in the left-hand side:

$$y_p'' - 4y_p' + 13y_p = 4Ae^{2x} - 8Ae^{2x} + 13Ae^{2x} = 9Ae^{2x} \quad [\mathbf{2p}]$$

so that to match to the right-hand side we should choose $A = 2$ so that $y_p(x) = 2\,e^{2x}$[**2p**]. Finally, the general solution to the non-homogeneous equation is given by the sum:

$$y_g(x) = (c_1 \cos 3x + c_2 \sin 3x + 2)e^{2x}, \qquad [\mathbf{1p}]$$

c) Find explicit solution to the following initial value problem:

$$y' = \frac{y + x}{\ln(y + x)} - 1\,, \quad y(0) = e \tag{5}$$

(13 points)

Solution. By introducing $z(x) = y + x$ [**1p**] we see that $z' = y' + 1$ [**1p**]and the ODE for z is reduced to the separable form

$$z' = \frac{z}{\ln z}\,, \quad [\mathbf{2p}]$$

Separating variables we have

$$\int \ln z \,\frac{dz}{z} = \int \ln z \, d(\ln z) = \frac{1}{2}\ln^2 z = x + C\,, \quad [\mathbf{3p}]$$

and solving for z we get

$$\ln z = \pm\sqrt{2(x+C)}, \quad \Rightarrow \quad z = e^{\pm\sqrt{2(x+C)}}\,, \quad [\mathbf{3p}]$$

which gives for $y = z - x$ the general solution:

$$y(x) = e^{\pm\sqrt{2(x+C)}} - x, \quad [\mathbf{1p}]$$

To find the solution to IVP (5) we notice: $y(0) = e^{\pm\sqrt{2C}}$ implying the plus sign and $2C = 1$ [**1p**]. Finally, the solution to IVP is given by

$$y(x) = e^{\sqrt{2x+1}} - x\,, \quad [\mathbf{1p}]$$