Queen Mary University of London
MTH5123 Differential Equations
January Exam Solutions 2022-2023

Question 1

(a) Find the general solution of the the second-order linear ODE

$$2y'' - 5y' - 3y = 0.$$

(6 marks)

- **Solution:** This is a second-order linear ODE with constant coefficients, thus we first write down the characteristic equation is $2\lambda^2 - 5\lambda - 3 = 0$ (2 marks). This equation has two real roots $\lambda_1 = -1/2, \lambda_2 = 3$ (2 marks). Hence, the general solution is $y_h = C_1 e^{-\frac{1}{2}x} + C_2 e^{3x}$ where $C_1, C_2 \in \mathbb{R}$ are arbitrary constants (2 marks).

(b) Find the general solution of the inhomogeneous second-order linear ODE

$$2y'' - 5y' - 3y = 10 \sin x.$$

(8 marks)

- **Solution:**
  Since the function $\sin x$ is not a solution to the homogeneous equation, we can use the educated guess method to find a particular solution. Therefore we look for the particular solution of the inhomogeneous equation in the form $y_p(x) = A \cos x + B \sin x$ (2 marks).
  Thus we have $y_p'(x) = -A \sin x + B \cos x$ (1 mark) and $y_p''(x) = -A \cos x - B \sin x$ (1 mark). Substituting these back into the inhomogeneous equation we get $(-2A - 5B - 3A) \cos x + (-2B + 5A - 3B) \sin x = -5(A+B) \cos x + 5(A-B) \cos x = 10 \sin x$. Therefore in order to satisfy this equation we should choose $A - B = 2$ and $A + B = 0$, hence $A = -B = 1$, and $y_p(x) = \cos x - \sin x$ (2 marks).
  Finally, using the results obtained in (a) we obtain that the general solution to the inhomogeneous equation is given by

  $$y_g(x) = y_h(x) + y_p(x) = C_1 e^{-\frac{1}{2}x} + C_2 e^{3x} + \cos x - \sin x$$

  where $C_1, C_2 \in \mathbb{R}$ are arbitrary constants. (2 marks).

(c) Find the solution to the Initial Value Problem

$$2y'' - 5y' - 3y = 10 \sin x, \quad y(0) = 4, \;\; y'(0) = 1.$$

(4 marks)

- **Solution:** Using the solution in (b), we impose the initial condition $y(0) = 4$ obtaining $y(0) = C_1 + C_2 + 1 = 4$ (1 marks). The first derivative of $y(x)$ is given by $y'(x) = -\frac{1}{2}C_1 e^{-\frac{1}{2}x} + 3C_2 e^{3x} - \sin x - \cos x$ (1 mark). Thus imposing the initial condition $y'(0) = 1$ we have $y'(0) = -\frac{1}{2}C_1 + 3C_2 - 1 = 1$ (1 marks). Solving these two linear equations for the constants C_1 and C_2, we obtain $C_1 = 2, C_2 = 1$. Therefore, the solution to this IVP is $y(x) = 2e^{-\frac{1}{2}x} + e^{3x} + \cos x - \sin x$ (1 marks).

Question 2

(a) Check whether the IVP

$$y' = \frac{x}{y-4}, \quad y(0) = 4.$$

satisfies the hypotheses of the Picard-Lindelöf theorem. **(6 marks)**

- **Solution:** The ODE in the considered IVP can be written as $y' = f(x,y)$ where $f(x,y) = \frac{x}{y-4}$ (1 mark). The initial condition imposes that the solution to the ODE passes through the point $(0,4)$ (1 mark). The first hypothesis of the Picard-Lindelöf theorem is that the function $f(x,y)$ is continuous in a (non vanishing) rectangular region centered around $(0,4)$, i.e. D is such that $|x| \leq A$ and $|y-4| \leq B$ for some $A > 0$, $B > 0$ (1 mark). We notice that the function $f(x,y) = 4/(y-4)$ diverges for $y \to 4$ (1 mark), therefore it is not continuous for $y = 4$, and hence is not continuous in any rectangular region D (1 mark). We conclude that the hypotheses of the Picard-Lindelöf theorem are not satisfied (1 mark).

(b) Find all the solutions of the IVP defined at point (a). Is this result in contradiction with the result obtained in point (a)? *Explain your answer.* **(8 marks)**
Solution: The ODE $y' = x/(y-4)$ is separable (1 mark). Using the separation of variables method we found that the anti-derivative $H(y) = \int (y-4)dy = \frac{(y-4)^2}{2}$ and the anti-derivative $F(x) = \int x dx = \frac{1}{2}x^2$ (2 marks). Hence the implicit solution of the ODE is $H(y) = F(x) + C$, i.e. $(y-4)^2 = x^2 + C$ where $C \in \mathbb{R}$ is an arbitrary constant (1 mark). The explicit solution is given by $y = 4 \pm \sqrt{x^2 + C}$ (1 mark). Imposing the initial condition we obtain $y(0) = 4 \pm \sqrt{C} = 4$, hence $C = 0$ leading to $y(x) = 4 \pm x$ (1 mark). It follows that the solution of the IVP is not unique (1 mark). This is consistent with the result obtained in point (a) as the Picard-Lindelöf does not guarantee the existence and uniqueness of the solution of this IVP (1 mark).

- Determine the smallest $b > 0$ such that the BVP

$$2y'' - 18y = \tanh(x), \quad y(0) = 0, y'(b) = 3,$$

does not have a unique solution. **(12 marks)**

Solution: This is an inhomogeneous BVP. According to the Theorem of the Alternative an inhomogeneous BVP has a unique solution if and only if its corresponding homogeneous BVP has a unique solution (2 marks). Therefore let us consider the corresponding homogeneous BVP given by

$$2y'' - 18y = 0, \quad y(0) = 0, y'(b) = 0.$$

(2 marks) The ODE is a second-order linear ODE with constant coefficients whose characteristic equation $2\lambda^2 - 18 = 0$ has complex conjugate roots $\lambda_1 = 3i$ and $\lambda_2 = -3i$ (1 mark). The general solution to this homogeneous ODE is given by $y_g(x) = A\cos 3x + B\sin 3x$ where $A, B \in \mathbb{R}$ are arbitrary constants (1 mark). The first derivative is $y_g'(x) = 3A\sin 3x - 3B\cos 3x$ (1 mark). Imposing the boundary conditions $y(0) = y'(b) = 0$ we obtain $y(0) = A = 0$ and $y'(b) = 3A\sin(3b) - 3B\cos(3b) = 0$ (1 mark). Using $A = 0$ the latter equation becomes $B\cos(3b) = 0$ (1 mark). Therefore if $\cos(3b) \neq 0$ we find a unique solution $A = B = 0$ while if $\cos(3b) = 0$, B can be arbitrary and the homogeneous BVP has infinite solutions (1 mark). We are interested in the case in which $A = B = 0$ is not the only solution. Therefore we need to consider the values for which $\cos(3b) = 0$, hence $3b \neq \frac{\pi}{2} + \pi n$ with $n \in \mathbb{Z}$ (1 mark). The smallest value of $b > 0$ for which both the inhomogeneous BVP and its corresponding homogeneous problem do not have a unique solution is $b = \frac{\pi}{2}$ (1 mark).

Question 3

(a) Find the general solution of the motion of a mass attached to the ceiling by a spring in presence of friction, i.e. solve the ODE

$$m\ddot{y} = mg - k(y - l) - \gamma\dot{y}.$$

with $m = 1, k = 3, \gamma = 2$, $g = 10$, $l = 5$ and y indicating the distance of the mass from the ceiling. **(8 marks)**

- **Solution:** We need to solve the ODE

$$\ddot{y} + 2\dot{y} + 3y = 25,$$

(1 marks). This is is an inhomogeneous second-order linear ODE with constant coefficients. We first find the general solution $y_h(t)$ of the corresponding homogeneous ODE

$$\ddot{y} + 2\dot{y} + 3y = 0$$

whose characteristic equation $\lambda^2+2\lambda+3 = 0$ has roots $\lambda_1 = -1+\sqrt{2}i$ $\lambda_2 = -1-\sqrt{2}i$. Therefore we obtain $y_h(t) = e^{-t}[A\cos(\sqrt{2}t) + B\sin(\sqrt{2}t)]$ with $A, B \in \mathbb{R}$ indicating arbitrary constants (3 marks). A particular solution of the inhomogeneous ODE can be obtained using the educated guess method by looking for a stationary solution $y_p(t) = d_0$. By observing that $\dot{y}_p(t) = \ddot{y}_p(t) = 0$ and plugging this solution into the inhomogeneous ODE we get $3d_0 = 25$, i.e. $d_0 = 25/3$ (3 marks). The general solution of the inhomogeneous ODE is

$$y_g(t) = \frac{25}{3} + e^{-t}[A\cos(\sqrt{2}t) + B\sin(\sqrt{2}t)$$

with $A, B \in \mathbb{R}$ indicating arbitrary constants (1 mark).

(b) Which is the limit $\lim_{t\to\infty} y(t)$ for the motion of the mass described in (a)? *Describe in words the asymptotic dynamical behaviour of the mass for $t \to \infty$.* **(4 marks)**

- **Solution:** The limit of $\lim_{t\to\infty} y_g(t) = \frac{25}{3}$ for every choice of the initial conditions (2 marks). Therefore asymptotically in time the mass reaches its static equilibrium at a distance from the ceiling given by $y = \frac{25}{3}$ (2 marks).

(c) Determine whether the differential equation

$$\frac{1}{2}y^2 + y\cos(x) + (yx + \sin(x) - e^y)\, y' = 0$$

is exact. If it is exact, find its general solution in explicit form. **(12 marks)**

- **Solution:** Denoting $P = \frac{1}{2}y^2 + y\cos(x), \quad Q = yx + \sin(x) - e^y$ (2 marks) we have $\frac{\partial P}{\partial y} = y + \cos(x) = \frac{\partial Q}{\partial x}$, so the equation is exact (2 marks). The general solution can be looked for in implicit form $F(x, y) = C$ (2 marks), where

$$F = \int P(x,y)\,dx = \int \left(\frac{1}{2}y^2x + y\cos(x)\right)\,dx = \frac{1}{2}y^2x + y\sin x + g(y),$$

(2 marks) where $g(y)$ is to be determined from the condition $Q = \frac{\partial F}{\partial y} = xy + \sin x + g'(y)$ (2 marks). We conclude that $g'(y) = -e^y$ so that $g(y) = -e^y + C_1$, with $C_1 \in \mathbb{R}$ arbitrary constant (2 marks). The solution in implicit form is given by $\frac{1}{2}y^2x + y\sin x - e^y = C$ (2 mark).

Question 4

Consider a system of two nonlinear first-order ODEs, where x and y are functions of the independent variable t:

$$\dot{x} = 2\tanh(x) - 2x\cos(y) + e^{x+3y} - 1, \quad \dot{y} = 3\cosh(x) - 3e^{xy} + \frac{1}{2}y + \frac{1}{2}\sin(x).$$

(a) Write down in matrix form of the type $\dot{\mathbf{X}} = A\mathbf{X}$ with $\mathbf{X} = (x, y)^\top$ the system obtained by linearisation of the above equations around the point $x = y = 0$. Specify the elements of the matrix A. (9 marks)

- **Solution** Expanding the relevant functions appearing in the system of ODEs close to the point $x = y = 0$ we obtain

$$\begin{aligned} \tanh(x) &= x + O(x^2), \\ \cos(y) &= 1 + O(y^2), \\ e^{x+3y} &= 1 + x + 3y + O((x+3y)^2), \\ \cosh(x) &= 1 + O(x^2), \\ e^{xy} &= 1 + O(xy), \\ \sin(x) &= x + O(x^2), \end{aligned} \tag{1}$$

(6 marks).
Keeping only the 0-order and first order terms in the expansion we obtain the linearised set of equations

$$\dot{x} = 2x - 2x + x + 3y = x + 3y, \quad \dot{y} = 3 - 3 + \frac{1}{2}y + \frac{1}{2}x = \frac{1}{2}(x + y),$$

(2 marks). Therefore the matrix A is given by

$$A = \begin{pmatrix} 1 & 3 \\ 1/2 & 1/2 \end{pmatrix},$$

(1 mark).

(b) Find the eigenvalues and eigenvectors of the matrix A obtained in (a). Write down the general solution of the linear system. **(8 marks)**

- **Solution:** The eigenvalues of the matrix A are found by solving $(1-\lambda)(1/2-\lambda) - 3/2 = 0$ giving $\lambda_1 = 2, \lambda_2 = -1/2$ (2 marks). The eigenvector corresponding to the eigenvalue $\lambda_1 = 2$ is $\mathbf{u}_1 = (3, 1)^\top$ (2 marks) and the eigenvector corresponding to the eigenvalue $\lambda_2 = -1/2$ is $\mathbf{u}_2 = (-2, 1)^\top$ (2 marks). The general solution of the system of ODEs is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{2t} + C_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-1/2t},$$

with $C_1, C_2 \in \mathbb{R}$ indicating arbitrary constants (2 marks).

(c) Which type of a fixed point is the equilibrium solution $x = y = 0$?
Sketch the phase portrait of the linear system. **(7 marks)**

- **Solution:** The fixed point $x = y = 0$ is a saddle. (2 marks) The phase portrait is given by Figure 1 (5 marks).

(d) Find the solution of the linear system corresponding to the initial conditions $x(0) = 1$, $y(0) = 0$. Determine the values $\lim_{t\to\infty} x(t)$ and $\lim_{t\to\infty} y(t)$. **(6 marks)**

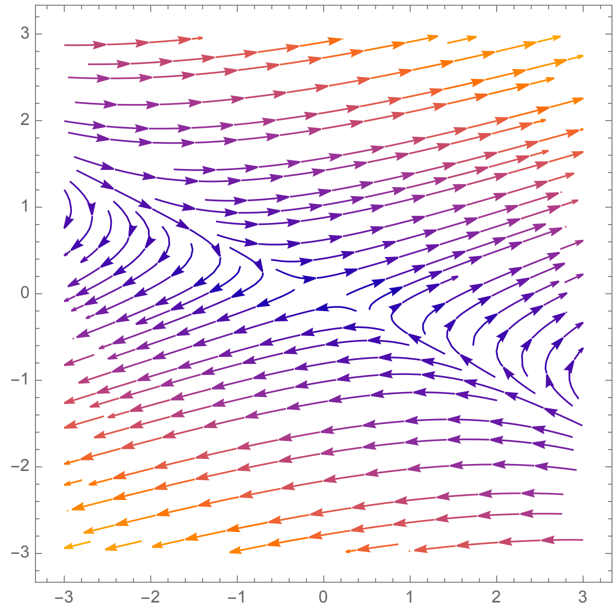

Figure 1: Streamplot of the linearised dynamical system. The equilibrium point is a saddle.

- **Solution:** By using the general solution obtained in (b) and imposing the initial conditions we obtain $3C_1 - 2C_2 = 1, C_1 + C_2 = 0$ (1 mark) which has solution $C_1 = -C_2 = 1/5$. (2 marks)Therefore the solution to this IVP is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{2t} - \frac{1}{5}\begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-1/2t}.$$

(1 mark) The limits are $\lim_{t\to\infty} x(t) = \infty$ and $\lim_{t\to\infty} y(t) = \infty$ (2 marks).