Stability of solutions of ODEs

Consider a system of ODEs and a given solution $Y_*(t)$ to the IVP with IC $Y_*(0)$

How different will be a solution $Y(t)$ to the same system of ODE and a slightly different initial condition?

Will the two solutions $Y(t)$ and $Y_*(t)$

- converge to each other
- remain close
- diverge

in the limit $t \to \infty$?

In order to address this question

- We formalise the notion of stability of solutions linear and non-linear autonomous systems of ODEs
- We will define
  1. Lyapunov stability
  2. Asymptotic stability

We consider the autonomous system of ODEs

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} f_1(y_1, y_2) \\ f_2(y_1, y_2) \end{pmatrix} \qquad (1)$$

including linear and non-linear systems of ODEs

Lyapunov stability

The concept "$Y_*(t)$ is Lyapunov stable if any solution $Y(t)$ remains close to $Y_*(t)$ as long as the two initial conditions $Y_*(0)$ and $Y(0)$ are sufficiently close to each other"

Definition A solution $Y_*(t)$ of the system of ODEs (1) corresponding to the I.C. $Y_*(0) = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$ is Lyapunov stable if

for an arbitrary $\varepsilon > 0$ we can find $\delta > 0$ such that

if another I.C. $Y(0) = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$ is chosen inside a circle of radius δ centered in $Y_*(0)$, if $|Y(0) - Y_*(0)| < \delta$

then, for any time $t > 0$ the solution $Y(t)$ corresponding to the I.C. $Y(0)$

1. exist

2. will satisfy $|Y(t) - Y_*(t)| < \varepsilon \quad \forall t$

Asymptotic stability

Definition

The solution $Y_*(t)$ to the system of ODEs (1) with I.C. $Y_*(0) = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$ is called

ASYMPTOTICALLY STABLE if

1. it is Lyapunov stable
2. there exist a $\delta > 0$ such that if

$$|Y(0) - Y_*(0)| < \delta \text{ then}$$

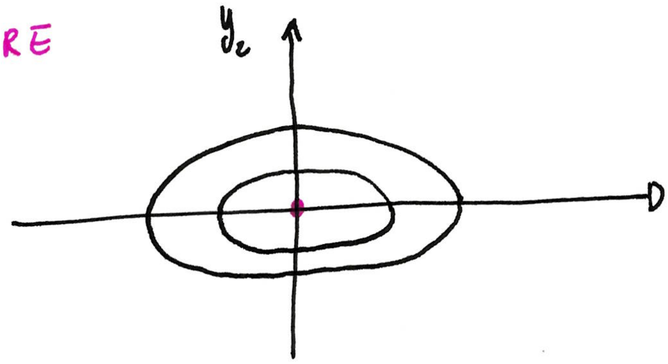
$$|Y(t) - Y_*(t)| \longrightarrow 0 \text{ as } t \to \infty$$

Lyapunov stability does not imply asymptotic stability

Example The equilibrium solution $Y(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ of a linear system of ODEs $\dot{Y} = AY$

displaying a centre as a phase portrait,

CENTRE



Trajectories satisfying

$$y_1^2 + 4y_2^2 = a^2$$

a. depends on the I.C.

$y_1(0) = y_2(0) = 0 \quad \Rightarrow a = 0 \qquad y_1^2 + 4y_2^2 = 0 \Rightarrow$

$y_1(t) = y_2(t) = 0$

Equilibrium point.

If

$y_1(0) = 1 \qquad y_2(0) = 0 \qquad y_1^2 + 4y_2^2 = 1$ at $t = 0$

but $y_1^2 + 4y_2^2 = 1 \quad \forall t$ because

$y_1^2 + 4y_2^2 = a = 1$ is the trajectory.

The trajectories remain close to the origin if a is small enough, but never reach the origin for $t \to \infty$

$Y_*(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is stable but not asymptotically stable