Invariant manifolds

<u>Example</u> Find the solution to the linear system of ODE

$$\dot{Y} = AY$$

where $A = \begin{pmatrix} -4 & 6 \\ -3 & 5 \end{pmatrix}$

of eigenvalues and eigenvectors

$\lambda_1 = 2$ $\quad$ $u_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2 = -1$ $\quad$ $u_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

The general solution is

$$Y(t) = D_1 e^{2t} u_1 + D_2 e^{-t} u_2$$

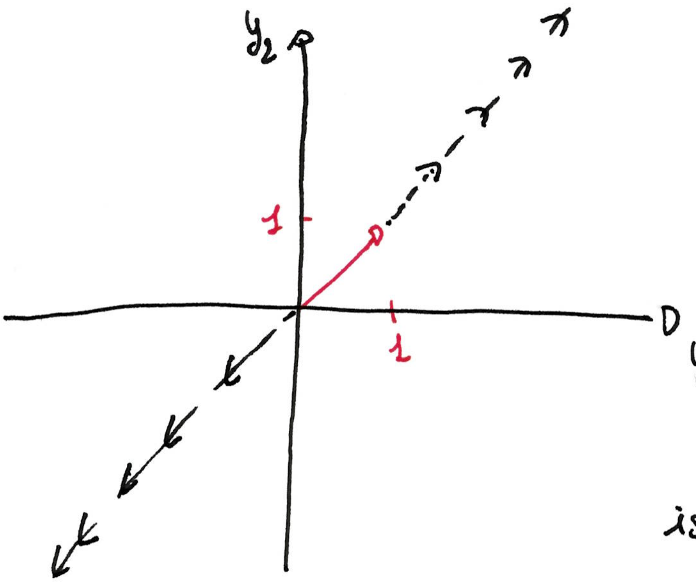
$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = D_1 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + D_2 e^{-t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

where $D_1, D_2 \in \mathbb{R}$ arbitrary constants.

Any $\overbrace{\text{IVP}}^{\text{solution to the}}$ with I.C. $Y(0) = k u_1 = k\begin{pmatrix}1\\1\end{pmatrix}$ with $k \in \mathbb{R}$

has a trajectory that moves along the direction u_1

and moves AWAY FROM THE ORIGIN.



$$Y(t) = k e^{2t} u_1$$

The direction defined by u_1

is an UNSTABLE INVARIANT MANIFOLD

$$\boxed{\text{This is generic for } \lambda_1 \in \mathbb{R},\ \lambda_1 > 0 \quad \lambda_1 \neq \lambda_2}$$

Unstable invariant manifolds are identified by the direction of the eigenvector associated to the real and positive eigenvalue, for $\lambda_1 \neq \lambda_2$

What happens if $Y(0) = k\,u_2$ with $k \in \mathbb{R}$?

Find solution to the IVP

$$\dot{Y} = AY \quad \text{with} \quad \text{I.C.} \quad Y(0) = k\,u_2$$

The general solution is

$$Y(t) = D_1 e^{\lambda_1 t} u_1 + D_2 e^{\lambda_2 t} u_2$$

Imposing I.C. $\quad Y(0) = D_1 u_1 + D_2 u_2 = k\,u_2$

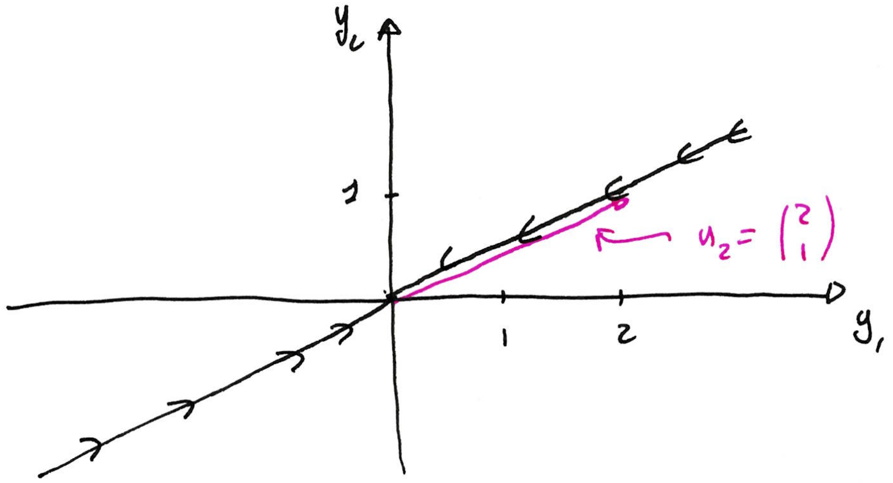
Rearranging $\quad \underbrace{D_1}_{=0} u_1 + \underbrace{(D_2 - k)}_{=0} u_2 = 0$

$$\begin{cases} D_1 = 0 \\ D_2 = k \end{cases} \Rightarrow \text{Solution to IVP}$$

$$Y(t) = k e^{\lambda_2 t} u_2 = k e^{-t} u_2$$

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = k e^{-t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \Rightarrow \begin{cases} y_1(t) = k e^{-t} \cdot 2 \\ y_2(t) = k e^{-t} \end{cases}$$

$$\boxed{y_1 = 2 y_2}$$ Trajectory



Trajectory

$y_2 = \frac{1}{2} y_1$

The direction determind by u_2 is a STABLE INVARIANT MANIFOLD

$$\begin{cases} y_1 = 2e^{-t} k \\ y_2 = k e^{-t} \end{cases}$$

If $k>0$ $\quad t \to \infty \quad y_1 \to 0^+ \quad y_2 \to 0^+$

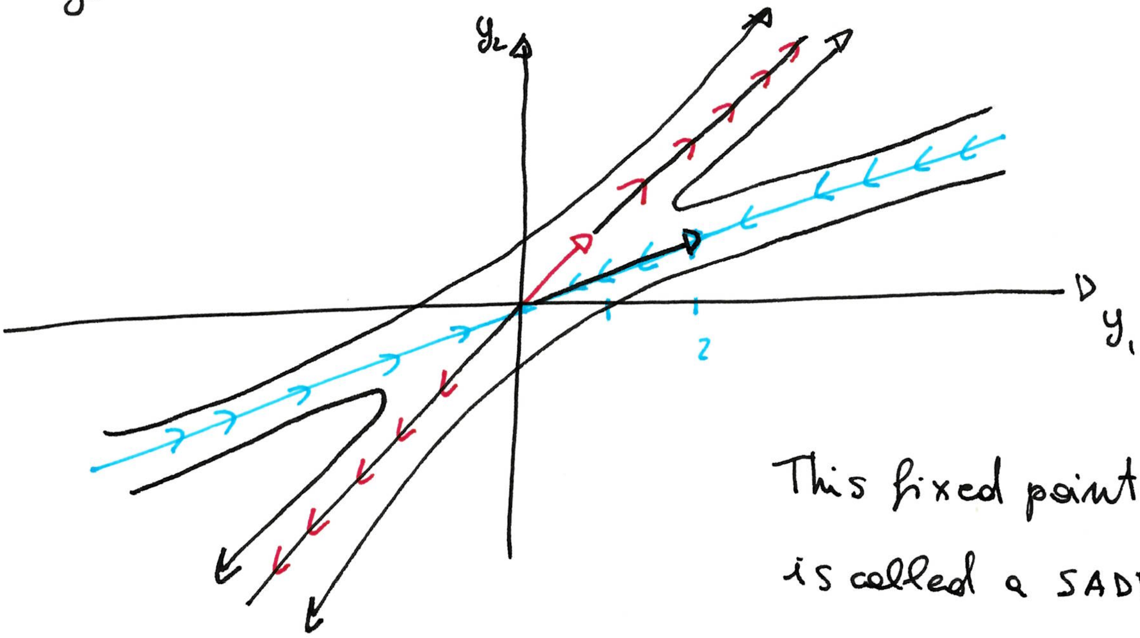
If $k<0$ $\quad t \to \infty \quad y_1 \to 0^- \quad y_2 \to 0^-$

All trajectories move along the direction of u_2 and move TOWARD THE ORIGIN

$$\boxed{\text{This is generic if } \lambda_2 \in \mathbb{R},\ \lambda_2 < 0 \quad \lambda_1 \neq \lambda_2}$$

Stable invariant manifolds are identified by the direction of the eigenvector associated to real and negative eigenvalues.

Let us put together the phase portrait of this dynamical system



This fixed point at (0,0) is called a SADDLE

This is the phase portrait when $\lambda_1, \lambda_2 \in \mathbb{R}$ $\lambda_1 > 0$, $\lambda_2 < 0$

λ_1, λ_2 with opposite sign.

Introduction to phase portraits

Consider the system of ODEs

$$\dot{Y} = AY$$

with A indicating a real 2×2 matrix with two distinct eigenvalues $\lambda_1 \neq \lambda_2$

What can we say about phase portraits?

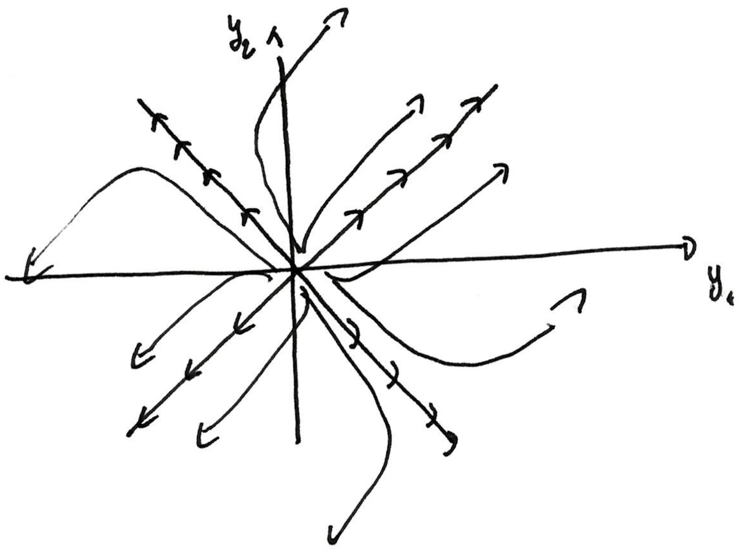
Knowing the eigenvalues we can already draw important conclusions but for the details we will need information about the eigenvectors as well.

(1) $\lambda_1 \neq \lambda_2$ $\quad \lambda_1, \lambda_2 \in \mathbb{R}$

a) $\lambda_1 > 0 \quad \lambda_2 > 0$

u_1 and u_2 are both UNSTABLE INVARIANT MANIFOLDS

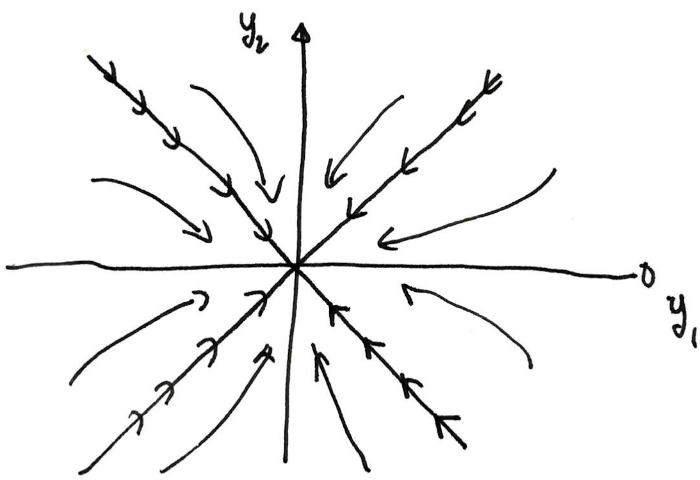
(0,0) is called an UNSTABLE NODE



b) $\lambda_1 < 0 \quad \lambda_2 < 0$

u_1 and u_2 are STABLE INVARIANT MANIFOLD

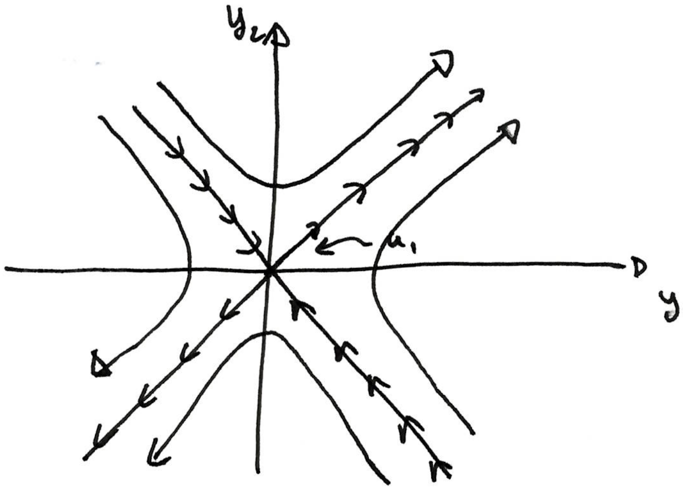
(0,0) is called a STABLE NODE



c) $\lambda_1 > 0 \quad \lambda_2 < 0$

$u_1 \to$ UNSTABLE INVARIANT MANIFOLD

$u_2 \to$ STABLE INVARIANT MANIFOLD



(0,0) is called a SADDLE.

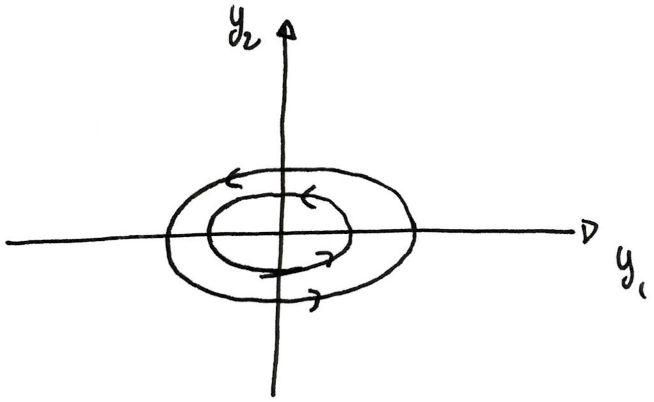
② $\lambda_1 = \alpha + i\beta \qquad \lambda_2 = \alpha - i\beta \quad , \alpha, \beta \in \mathbb{R} \quad \beta \neq 0$

Complex conjugate eigenvalues.

$$e^{\lambda_1 t} = e^{\alpha t}(\cos \beta t + i \sin \beta t)$$

a) If $\alpha = 0$

(0,0) is a CENTRE

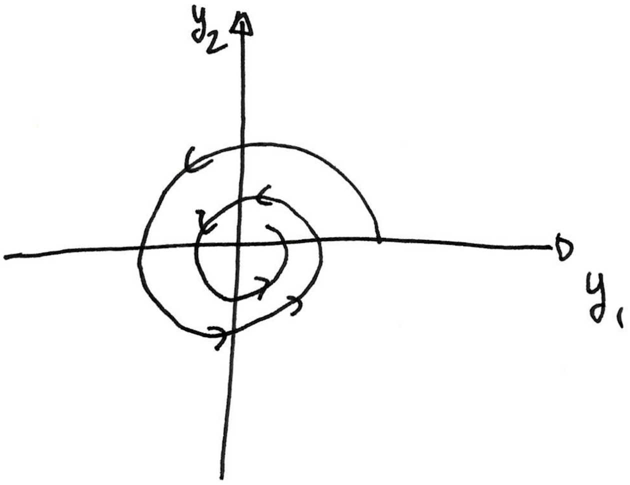


All trajectories are periodic

Periodic orbits

If $\alpha < 0$

(0,0) is a STABLE FOCUS

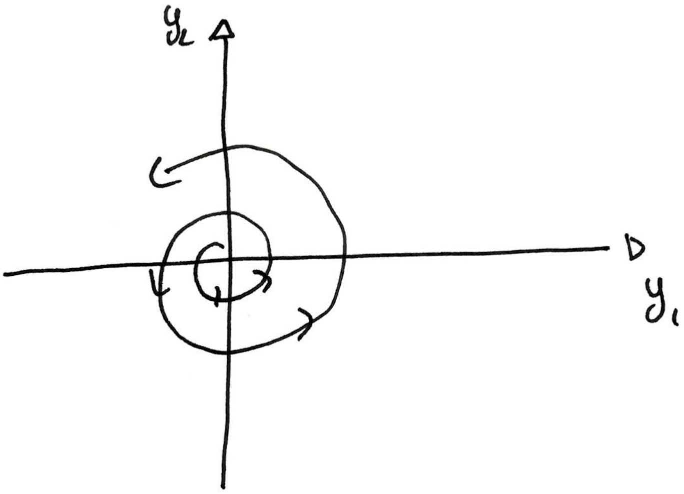


The phase portrait is a SPIRAL IN

All trajectories go to the origin.

If $\alpha > 0$

(0,0) is an USTABLE FOCUS



The phase portrait is a SPIRAL OUT

All trajectories go away from the origin.