Formative assessment week 5

1) Solve the ODE

$$y'' + 7y' + 6y = 10 \sin 2x \qquad (1)$$

This is an inhomogeneous ODE

The corresponding homogeneous ODE is

$$y'' + 7y' + 6y = 0 \qquad (2)$$

whose characteristic equation is

$$\Pi_2(\lambda) = \lambda^2 + 7\lambda + 6 = 0$$

of roots
$$\lambda = \frac{-7 \pm \sqrt{49-24}}{2} = \frac{-7 \pm \sqrt{25}}{2} = \frac{-7 \pm 5}{2} = \begin{cases} -6 \\ -1 \end{cases}$$

$\lambda_1 = -6 \quad \lambda_2 = -1$

The general solution to the homogeneous ODE (2) is

$$y_h(x) = c_1 e^{-6x} + c_2 e^{-x}$$ where c_1, c_2 are arbitrary constants

$c_1, c_2 \in \mathbb{R}$

W

Since the ODE (1) is of the type

$p(x) = 10 \quad (k=0)$

$a = 2$

$$a_2 y'' + a_1 y' + a_0 y = p(x) \sin a x$$

$p(x)$ is a polynomial of degree $k = 0$ ✓

$\sin 2x$ or $\cos 2x$ are not solutions to (1) ✓

We can use the educated guess method.

Therefore we look for a particular solution of the type

$$y_p(x) = Q(x) \left[A' \cos 2x + B' \sin 2x \right]$$

where $Q(x)$ is a polynomial of degree $k = 0$

$\Rightarrow Q(x) = d_0$, d_0, A', B' are real parameters.

$$y_p(x) = d_0 \left[A' \cos 2x + B' \sin 2x \right]$$

$$y_p(x) = \left[A \cos 2x + B \sin 2x \right] \qquad A = d_0 A' \quad B = d_0 B'$$

Let us substitute $y_p(x)$ into the ODE (1)

To this end we calculate $y_p'(x)$, $y_p''(x)$

$$y_p'(x) = \left[-2A \sin 2x + 2B \cos 2x \right]$$

$$y_p''(x) = \left[-4A \cos 2x - 4B \sin 2x \right]$$

Inserting $y_p(x)$, $y_p'(x)$, $y_p''(x)$ into Eq (1)

[which is given by $y'' + 7y' + 6y = 10 \sin 2x$]

$$\left[-4A\cos 2x - 4B\sin 2x\right] + 7\left[-2A\sin 2x + 2B\cos 2x\right] +$$

$$+6\left[A\cos 2x + B\sin 2x\right] = 10\sin 2x$$

Rearranging

$$\cos 2x\,(-4A + 14B + 6A) + \sin 2x\,(-4B - 14A + 6B) = 10\sin 2x$$

$$\begin{cases} -4A + 14B + 6A = 0 \\ -4B - 14A + 6B = 10 \end{cases} \qquad \begin{cases} 2A + 14B = 0 \\ 2B - 14A = 10 \end{cases}$$

$$\begin{cases} A = -7/10 \\ B = \frac{1}{10} \end{cases} \qquad \boxed{y_p = -\frac{7}{10}\cos 2x + \frac{1}{10}\sin 2x}$$

The general solution to ODE (1) is

$$y_g(x) = \underbrace{c_1 e^{-6x} + c_2 e^{-x}}_{y_h(x)} \underbrace{- \frac{7}{10}\cos 2x + \frac{1}{10}\sin 2x}_{y_p(x)}, \quad c_1, c_2 \in \mathbb{R} \text{ arbitrary constants}$$