MTH4104 **Introduction to Algebra**

Week 10 tutorial questions **w/c 28 March 2022**

This sheet contains questions for you to work through in your tutorial, singly or in a group.

It's important to work through lots of questions for practice. Remember that mathematics is not a spectator sport! If you want more questions, look at the "Extra questions" sheets on QMPlus.

Question 1 As before, this question is to encourage you to try filling in a piece of a proof skipped in lecture.

Let R be a ring. Prove the left distributive law for $\mathrm{M}_2(R)$.

Solution Write $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ and $C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$ where the coefficients

$$a_{11}, a_{12}, a_{21}, a_{22}, b_{11}, b_{12}, b_{21}, b_{22}, c_{11}, c_{12}, c_{21}, c_{22}$$

are elements of the ring R.

We have $B + C = \begin{pmatrix} b_{11}+c_{11} & b_{12}+c_{12} \\ b_{21}+c_{21} & b_{22}+c_{22} \end{pmatrix}$. Hence

$$\begin{aligned} A(B+C) &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11}+c_{11} & b_{12}+c_{12} \\ b_{21}+c_{21} & b_{22}+c_{22} \end{pmatrix} = \\ &= \begin{pmatrix} a_{11}(b_{11}+c_{11})+a_{12}(b_{21}+c_{21}) & a_{11}(b_{12}+c_{12})+a_{12}(b_{22}+c_{22}) \\ a_{21}(b_{11}+c_{11})+a_{22}(b_{21}+c_{21}) & a_{21}(b_{12}+c_{12})+a_{22}(b_{22}+c_{22}) \end{pmatrix}. \end{aligned}$$

The distributive law is valid in the ring R. Applying it to each entry in this matrix twice (i.e. expanding all the brackets, whilst being careful *not* to write (say) $b_{21}a_{12}$ instead of $a_{12}b_{21}$), we get

$$A(B+C) = \begin{pmatrix} a_{11}b_{11}+a_{11}c_{11}+a_{12}b_{21}+a_{12}c_{21} & a_{11}b_{12}+a_{11}c_{12}+a_{12}b_{22}+a_{12}c_{22} \\ a_{21}b_{11}+a_{21}c_{11}+a_{22}b_{21}+a_{22}c_{21} & a_{21}b_{12}+a_{21}c_{12}+a_{22}b_{22}+a_{22}c_{22} \end{pmatrix}.$$

On the other hand, we have

$$\begin{aligned} AB+AC &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \\ &= \begin{pmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \end{pmatrix} + \begin{pmatrix} a_{11}c_{11}+a_{12}c_{21} & a_{11}c_{12}+a_{12}c_{22} \\ a_{21}c_{11}+a_{22}c_{12} & a_{21}c_{12}+a_{22}c_{22} \end{pmatrix} = \\ &= \begin{pmatrix} a_{11}b_{11}+a_{12}b_{21}+a_{11}c_{11}+a_{12}c_{21} & a_{11}b_{12}+a_{12}b_{22}+a_{11}c_{12}+a_{12}c_{22} \\ a_{21}b_{11}+a_{22}b_{21}+a_{21}c_{11}+a_{22}c_{21} & a_{21}b_{12}+a_{22}b_{22}+a_{21}c_{12}+a_{22}c_{22} \end{pmatrix}. \end{aligned}$$

This is equal to $A(B+C)$ by the commutativity of addition in R.

Question 2 Let R be a ring. Prove that the set

$$\left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in R \right\}$$

of 2×2 matrices over R whose lower left entry equals zero is a ring, with the usual addition and multiplication of matrices. You may assume that $\mathrm{M}_2(R)$ is a ring.

Solution Let S be the set in the question. We will show S is a ring by proving the ring axioms.

The key feature of our situation is that S is defined as a subset of the ring $M_2(R)$. As emphasised in lecture, this is a situation where the closure laws are particularly important. On the other hand, a number of the ring axioms for S will automatically be true by virtue of the fact that elements of S are elements of $M_2(R)$, and the operations work the same way. For brevity, I will just say "automatic" below for these laws.

Closure for $+$. We must show that the sum

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} + \begin{pmatrix} d & e \\ 0 & f \end{pmatrix}$$

of two elements of S is also in S. This sum is

$$\begin{pmatrix} a+d & b+e \\ 0 & c+f \end{pmatrix},$$

which has the requisite zero in the lower-left corner, and is therefore in S.

Closure for $\cdot$. Now we must compute the product

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \cdot \begin{pmatrix} d & e \\ 0 & f \end{pmatrix}$$

and show that it is in S. The product is

$$\begin{pmatrix} a \cdot d + b \cdot 0 & a \cdot e + b \cdot f \\ 0 \cdot d + c \cdot 0 & 0 \cdot e + c \cdot f \end{pmatrix} = \begin{pmatrix} ad & ae + bf \\ 0 & cf \end{pmatrix},$$

which again is in S by virtue of the lower-left entry being zero.

Associativity for $+$. Automatic.

Associativity for $\cdot$. Automatic.

Identity for $+$. We note that the additive identity matrix of $M_2(R)$

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

is in the set S. Then the fact that it still behaves as an identity for the addition in S is automatic.

Inverses for $+$. Given a matrix

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

in S, the matrix

$$\begin{pmatrix} -a & -b \\ 0 & -c \end{pmatrix},$$

which is its additive inverse within $M_2(R)$, lies in S. Then the fact that it's still the inverse in S is automatic.

Commutativity for $+$. Automatic.

Distributivity. Automatic.

Question 3 Label the corners of a regular pentagon 1 through 5 in order. Each of the symmetries of the pentagon—reflections, rotations, and the identity (doing nothing)—gives a permutation in S_5 describing how it moves the corners.

(a) Write down the set of all these permutations.

(b) Is your set closed under composition?

Solution (a) I labelled my corners going anticlockwise. If you did it clockwise, what I'm about to say is still true if you replace the word "anticlockwise" with "clockwise".

We have three kinds of symmetries.

- The identity, doing nothing. This is $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$.

  (You might be bothered that I count the identity as a symmetry, but if I want all the symmetries to form a group, I have to do!)

- Rotations through the centre. There are four, by angles $2\pi/5$, $4\pi/5$, $6\pi/5$, $8\pi/5$ anticlockwise. The rotation through $4\pi/5$, for instance, is the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 1 & 2 \end{pmatrix}$.

- Reflections. There are five, one across each line joining a corner to the midpoint of the opposite side. If I pick the reflection line through corner 1, for instance, I get the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 4 & 3 & 2 \end{pmatrix}$.

Altogether, my set is

$$\left\{ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 4 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 5 & 4 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 5 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 2 & 1 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 \end{pmatrix} \right\}.$$

(b) $\boxed{\text{Yes}}$, this set is closed under composition. You can check this by working out the composition of all 10^2 pairs of permutations in the set.

Here's a more conceptual, but intentionally vague, argument. A symmetry of the pentagon P is some sort of motion that leaves the pentagon looking the same as it did before. Let's think of these symmetries as functions. Then if f and g are two symmetries of the pentagon, we have $f(P) = P$ and $g(P) = P$, so

$$(f \circ g)(P) = f(g(P)) = f(P) = P$$

and $f \circ g$ is also a symmetry.

Exercise for the reader: make the above vague argument rigorous! For example, what should the domain and codomain of f and g be? How do f and g relate to the permutations?

Question 4 A *permutation matrix* in $\mathrm{M}_n(\mathbb{R})$ is a matrix so that, in each row and column, there is exactly one 1, and all other entries are 0. For example, an identity matrix of any size is a permutation matrix.

(a) Pick some $n \geq 3$ and write down two more permutation matrices in $\mathrm{M}_n(\mathbb{R})$, other than the identity matrix. Then compute their product.

(b) Why is the name "permutation matrix" appropriate?

Solution (a) There are many correct answers to this part, of course. Here are two arbitrary examples of permutation matrices in $\mathrm{M}_4(\mathbb{R})$:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

These two matrices have product

$$AB = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

another permutation matrix! Indeed, the product of any two permutation matrices will be a permutation matrix, so even if you picked different examples you should have got a permutation matrix as product too.

Working out the product isn't as painful as the size of the matrix suggests, because we're only multiplying and adding 0s and 1s. For example, the computation for the top left entry of AB is

$$0 \cdot 0 + 0 \cdot 1 + 1 \cdot 0 + 0 \cdot 0 = 0.$$

(b) The first relationship is that there is one permutation matrix in $\mathrm{M}_n(\mathbb{R})$ for each permutation of $\{1, \ldots, n\}$. There are $n!$ of each in total. Two sets of the same size have bijections between them, and you can make a nice bijection (let's call it ϕ) between permutations and permutation matrices as follows. To make a permutation matrix, we have to put a 1 somewhere in each column. So given a permutation $f \in S_n$, let's put the 1 in the ith column of $\phi(f)$ into the $f(i)$th row. To say this another way,

$$\phi(f)_{ij} = \begin{cases} 1 & \text{if } i = f(j) \\ 0 & \text{otherwise.} \end{cases}$$

I'll do an example. For the matrix A above,

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

the 1st column has its 1 in row 4, the 2nd column in row 2, the 3rd column in row 1, and the 4th column in row 3. So

$$\phi\left(\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}\right) = A.$$

But the key thing I wanted you to notice is that permutation matrices multiply "the same way" as permutations compose. What I mean by that is, if f and g are two permutations in S_n, then

$$\phi(f \circ g) = \phi(f) \cdot \phi(g),$$

with permutation composition on the left "translated" into matrix multiplication on the right. In my example above, $f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}$, $g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$, and $f \circ g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}$. Check that these match the matrices A, B, and AB respectively.

Why is this equation true? The entry at position i, j of the left hand side is

$$\phi(f \circ g)_{ij} = \begin{cases} 1 & \text{if } i = f(g(j)) \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

On the right hand side, we have

$$\big(\phi(f) \cdot \phi(g)\big)_{ij} = \sum_{k=1}^{n} \phi(f)_{ik} \phi(g)_{kj}.$$

There's just one k for which $\phi(f)_{ik} = 1$, namely $k = f^{-1}(i)$, and just one k for which $\phi(g)_{kj} = 1$, namely $k = f(j)$. If those two ks are different, then the sum will be $1 \cdot 0$ plus $0 \cdot 1$ plus many copies of $0 \cdot 0$, summing to 0. But if the two ks are the same — in other words, if $f^{-1}(i) = f(j)$ — then the sum has $1 \cdot 1$ as its kth term, plus a load of $0 \cdot 0$ summands, adding up to 1. That is,

$$\big(\phi(f) \cdot \phi(g)\big)_{ij} = \begin{cases} 1 & \text{if } f^{-1}(i) = g(j) \\ 0 & \text{otherwise.} \end{cases}.$$

And this right hand side is the same as in Equation (1).

By the way, you might have come up with the "other" natural bijection, that gives matrices which are the transpose of mine. This is the function ψ so that $\psi(f)_{ij}$ equals 1 if $f(i) = j$ and 0 otherwise. In that case the relationship between multiplication and composition would have been

$$\psi(f \circ g) = \psi(g) \cdot \psi(f).$$