MTH4104 **Introduction to Algebra**

Week 9 tutorial questions **w/c 21 March 2022**

This sheet contains questions for you to work through in your tutorial, singly or in a group.

It's important to work through lots of questions for practice. Remember that mathematics is not a spectator sport! If you want more questions, look at the "Extra questions" sheets on QMPlus.

1 What are the quotient and remainder when x^5 is divided by $x^2+(1+\mathrm{i})x+\mathrm{i}$ in $\mathbb{C}[x]$?

2 Let R be the relation on the set $\mathbb{R}[x]$ defined by

$$fRg \text{ if and only if } x^2+1 \text{ divides } g-f$$

for all $f,g \in \mathbb{R}[x]$.

(a) Prove that R is an equivalence relation.

(b) Prove that, in every equivalence class of R, there is exactly one polynomial h such that $h=0$ or $\deg h \leq 1$.

(c) Let $f,g \in \mathbb{R}[x]$ be polynomials such that fRg. Prove that $f(\mathrm{i}) = g(\mathrm{i})$ as complex numbers. From this point of view, what is special about the polynomial h from part (b)?

3 (a) Give an example of a ring R and a polynomial $f \in R[x]$ such that the number of solutions $\alpha \in R$ to $f(\alpha)=0$ is greater than $\deg f$.

[Hint: I've shown you one in lectures in a different context.]

(b) If the proofs in Sections 4.3 and 4.5 of the lecture notes were applicable to your ring R from part (a), they would imply that $f(\alpha)=0$ could have at most $\deg f$ solutions. If you try to apply these proofs to R, where do they go wrong?

Recall that, if R is a ring, $\mathrm{M}_n(R)$ is our notation for the ring of $n \times n$ square matrices with elements of R as entries.

4 If x and y are elements of a ring R such that $xy = yx$, we say that x and y *commute*, or that x *commutes with* y.

Find all matrices in $\mathrm{M}_2(\mathbb{R})$ which commute with $\begin{pmatrix} 3 & 4 \\ -2 & -3 \end{pmatrix}$.

5 Let R be a ring, and n a natural number. Describe the rings $\mathrm{M}_n(R[x])$ and $\mathrm{M}_n(R)[x]$. Give an example of an element of each ring. Explain how these two rings relate to each other.