Queen Mary University of London

MTH4104 **Introduction to Algebra**

Week 10 coursework **due 4 April 2022**

You are to write up a careful and professionally presented solution to the question below. This is to be submitted on QMPlus as a single PDF or JPEG file by 12:00 noon, Monday 4 April 2022.

Question to submit For a natural number n, let

$$C_n = \{(a_{ij}) \in M_n(\mathbb{R}) : a_{ij} = a_{kl} \text{ for all } i,j,k,l \text{ such that } j-i \equiv_n l-k\}.$$

(a) Exactly one of the two matrices below is an element of C_3. Which one? [1 mark]

$$A_1 = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{pmatrix}$$

(b) Let A be the matrix you chose in part (a). Compute A^2. Check that $A^2 \in C_3$, and write out a brief explanation (one or two sentences is enough) explaining how you checked. [2 marks]

(c) Give a complete proof that C_3 is a ring. You may assume that $M_3(\mathbb{R})$ is a ring. Hint: make as much use of that assumption as possible in your proof. [7 marks]

Solution

Short model answer (a) $\boxed{A_2}$.

(b)

$$A^2 = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{pmatrix}^2 = \begin{pmatrix} 4 & 4 & 1 \\ 1 & 4 & 4 \\ 4 & 1 & 4 \end{pmatrix}.$$

A matrix is in C_3 if each of the three "wrap-around diagonals" in boxes below contains the same real number repeated three times. This is true for A^2.

$$\begin{pmatrix} \boxed{4} & 4 & 1 \\ 1 & \boxed{4} & 4 \\ 4 & 1 & \boxed{4} \end{pmatrix} \quad \begin{pmatrix} 4 & \boxed{4} & 1 \\ 1 & 4 & \boxed{4} \\ \boxed{4} & 1 & 4 \end{pmatrix} \quad \begin{pmatrix} 4 & 4 & \boxed{1} \\ \boxed{1} & 4 & 4 \\ 4 & \boxed{1} & 4 \end{pmatrix}$$

(c) We must prove all of the ring axioms.

Additive closure law Let $A = \begin{pmatrix} a_0 & a_1 & a_2 \\ a_2 & a_0 & a_1 \\ a_1 & a_2 & a_0 \end{pmatrix}$ and $B = \begin{pmatrix} b_0 & b_1 & b_2 \\ b_2 & b_0 & b_1 \\ b_1 & b_2 & b_0 \end{pmatrix}$ be two arbitrary elements of C_3. Then

$$A+B = \begin{pmatrix} a_0+b_0 & a_1+b_0 & a_2+b_2 \\ a_2+b_2 & a_0+b_0 & a_1+b_0 \\ a_1+b_1 & a_2+b_2 & a_0+b_0 \end{pmatrix}$$

is in C_3, as we can see because the three wrapping diagonals each contain the same sum three times.

Additive associative law True for C_3 because it is true for all elements of $\mathrm{M}_3(\mathbb{R})$ and $C_3 \subseteq \mathrm{M}_3(\mathbb{R})$.

Additive identity law The additive identity element is

$$0_{C_3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0_{\mathrm{M}_3(\mathbb{R})}.$$

This is in C_3 by inspection (entries on wrapping diagonals are the same). For any $A \in C_3, A + 0_{C_3} = 0_{C_3} + A = A$ is true because it is true in $\mathrm{M}_3(\mathbb{R})$ and $C_3 \subseteq \mathrm{M}_3(\mathbb{R})$.

Additive inverse law Let $A \in C_3$ be as above. Then

$$-A = \begin{pmatrix} -a_0 & -a_1 & -a_2 \\ -a_2 & -a_0 & -a_1 \\ -a_1 & -a_2 & -a_0 \end{pmatrix}$$

is in C_3 by inspection. The equation $A + (-A) = (-A) + A = 0$ is true in C_3 because it is true in $\mathrm{M}_3(\mathbb{R})$.

Additive commutative law True for C_3 because it is true for all of $\mathrm{M}_3(\mathbb{R})$.

Multiplicative closure law Let $A, B \in C_3$ be as above. Then

$$\begin{aligned} AB &= \begin{pmatrix} a_0b_0 + a_1b_2 + a_2b_1 & a_0b_1 + a_1b_0 + a_2b_2 & a_0b_2 + a_1b_1 + a_2b_0 \\ a_2b_0 + a_0b_2 + a_1b_1 & a_2b_1 + a_0b_0 + a_1b_2 & a_2b_2 + a_0b_1 + a_1b_0 \\ a_1b_0 + a_2b_2 + a_0b_1 & a_1b_1 + a_2b_0 + a_0b_2 & a_1b_2 + a_2b_1 + a_0b_0 \end{pmatrix} \\ &= \begin{pmatrix} a_0b_0 + a_1b_2 + a_2b_1 & a_0b_1 + a_1b_0 + a_2b_2 & a_0b_2 + a_1b_1 + a_2b_0 \\ a_0b_2 + a_1b_1 + a_2b_0 & a_0b_0 + a_1b_2 + a_2b_1 & a_0b_1 + a_1b_0 + a_2b_2 \\ a_0b_1 + a_1b_0 + a_2b_2 & a_0b_2 + a_1b_1 + a_2b_0 & a_0b_0 + a_1b_2 + a_2b_1 \end{pmatrix} \end{aligned}$$

is in C_3. (The second matrix is just the first one rearranged to make it clear that the entries that should match do.)

Multplicative associative law True for C_3 because it is true for all of $\mathrm{M}_3(\mathbb{R})$.

Distributive law True for C_3 because it is true for all of $\mathrm{M}_3(\mathbb{R})$.

Commentary The usual name given to C_n is the ring of $n \times n$ *circulant* matrices (with entries in $\mathbb{R}$). It shouldn't surprise you to learn that C_n is a ring for all n, and that this remains true if you change $\mathbb{R}$ in the question to some other ring.

Parts (a) and (b) of the question were mostly exercises in understanding set notation. The parts of the proof for part (c) where you couldn't just invoke the axioms for $\mathrm{M}_3(\mathbb{R})$ and be finished were those parts where you had to show some matrix was an element of C_3, so I wanted to make sure it was completely clear what it entails to show that. Also, in part (c), you had to take various arbitrary elements of C_3, so parts (a,b) were meant to guide you to an easier way to write down such an arbitrary element, as I used in my model solution.

Part (c) is based on the same idea as part (b) from the week 8 coursework, but more explicit. In a sentence: if S is a subset of a ring R and you want to know whether S is a ring (with the same plus and times operations), you just have to check the "there exists" parts of the axioms. Aside from that, the axioms for S are nothing but special cases of the axioms of R, applied to those elements of R that happen to be in S.