MTH4104 **Introduction to Algebra**

Week 8 coursework **due 21 March 2022**

You are to write up a careful and professionally presented solution to the question below. This is to be submitted on QMPlus as a single PDF or JPEG file by 12:00 noon, Monday 21 March 2022.

Question to submit Let $\mathbb{R}[[x]]$ be the set of all expressions

$$a = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{i=0}^{\infty} a_i x^i$$

where $a_i \in \mathbb{R}$ for all nonnegative integers i. Informally, an element of $\mathbb{R}[[x]]$ is like a polynomial except that it can have infinitely many terms.

(a) Carefully write down definitions of addition and multiplication operations for $\mathbb{R}[[x]]$, analogous to the definitions for $\mathbb{R}[x]$ in the notes. Given $a, b \in \mathbb{R}[[x]]$, your definitions should indicate what each coefficient of the sum $a+b$ and product ab is. [2 marks]

(b) Let $f = a_0 + a_1 x + \cdots + a_n x^n$ be a polynomial. I can treat f as an element of $\mathbb{R}[[x]]$ by defining $a_{n+1}, a_{n+2}, \ldots$ all to equal 0. This shows that $\mathbb{R}[x] \subseteq \mathbb{R}[[x]]$.

If you had already proved that $\mathbb{R}[[x]]$ was a ring, how could you use this fact to help you prove $\mathbb{R}[x]$ is a ring? [2 marks]

(c) Let $a \in \mathbb{R}[[x]]$ with $a_0 \neq 0$. Prove that a has a multiplicative inverse in $\mathbb{R}[[x]]$. You may assume that the multiplicative identity element in $\mathbb{R}[[x]]$ is

$$1_{\mathbb{R}[[x]]} = 1 + 0x + 0x^2 + 0x^3 + \cdots,$$

and that multiplication in $\mathbb{R}[[x]]$ is commutative. [6 marks]

[Hint. If $ab = 1_{\mathbb{R}[[x]]}$, equate coefficients and solve for $b_0, b_1, b_2, \ldots$ in turn.]

Solution

Short model answer (a) For $a = \sum_{i=0}^{\infty} a_i x^i$ and $b = \sum_{i=0}^{\infty} b_i x^i$ in $\mathbb{R}[[x]]$,

$$a + b = \sum_{i=0}^{\infty} (a_i + b_i) x^i,$$

$$ab = \sum_{i=0}^{\infty} (a_i b_0 + a_{i-1} b_1 + \cdots + a_0 b_i) x^i.$$

(b) The associative, commutative, and distributive laws for $\mathbb{R}[x]$ are just special cases of the same laws for $\mathbb{R}[[x]]$. For example, the additive commutative law says that, for all $a, b \in \mathbb{R}[[x]]$, $a+b = b+a$. Since all elements of $\mathbb{R}[x]$ are elements of $\mathbb{R}[[x]]$, this implies the additive commutative law for $\mathbb{R}[x]$.

(c) Define $b_0 = a_0^{-1}$, and for each $i \geq 1$ in increasing order, define

$$b_i = a_0^{-1} \cdot (-a_i b_0 - a_{i-1} b_1 - \cdots - a_1 b_{i-1}).$$

Let $b = \sum_{i=0}^{\infty} b_i x^i$. Using the definition of multiplication from part (a), the constant coefficient of ab is

$$a_0 b_0 = a_0 a_0^{-1} = 1$$

and, for $i \geq 1$, the ith coefficient of ab is

$$\begin{aligned}
&a_i b_0 + \cdots + a_1 b_{i-1} + a_0 b_i \\
&= a_i b_0 + \cdots + a_1 b_{i-1} + a_0 \cdot a_0^{-1}(-a_i b_0 - \cdots - a_1 b_{i-1}) \\
&= a_i b_0 + \cdots + a_1 b_{i-1} - a_i b_0 - \cdots - a_1 b_{i-1} \\
&= 0.
\end{aligned}$$

Compiling all these computations of coefficients, we have

$$ab = 1 + 0x + 0x^2 + \cdots = 1_{\mathbb{R}[[x]]}.$$

Because the multiplication is commutative, $ba = 1_{\mathbb{R}[[x]]}$ as well. So b is the multiplicative inverse of a.

Discussion. For any ring R, you can define $R[[x]]$ in the same way. If R is a field, part (c) will still be true. I asked the question just for $R = \mathbb{R}$ to try to make it a little less intimidating.

$R[[x]]$ is called the ring of *formal power series* with coefficients in R. You have met power series in your Calculus modules. The adjective “formal” has the same function here as it does in “formal symbol”: we’re not thinking e.g. of convergence of these power series, just treating them as objects that we can do algebraic manipulation to. (I left this term out of the question to reduce the temptation to google it.)

(a) I used sigma notation for sums because they’re more compact. There’s no problem if you prefer to write, for example, $a = a_0 + a_1 x + a_2 x^2 + \cdots$ instead of $a = \sum_{i=0}^{\infty} a_i x^i$.

(b) I could have mentioned other laws too in my answer, for instance the identity and inverse laws. For the additive identity law, you have to check that $0_{\mathbb{R}[[x]]}$ is an element of $\mathbb{R}[x]$, and for the additive inverse law, you have to check that if $f \in \mathbb{R}[x]$ then $-f \in \mathbb{R}[x]$, but after you do that, the equations for $\mathbb{R}[x]$ are the same equations already proved for $\mathbb{R}[[x]]$. In your answer, I’m happy even if you only mentioned one law, and wrote less than I did, as long as you were clear.

It’s critical to the logic of this question that if you add or multiply two polynomials $f, g \in \mathbb{R}[x]$ using the definitions for $\mathbb{R}[x]$ or the definitions for $\mathbb{R}[[x]]$, you get the same answer. Otherwise, the same axiom for $\mathbb{R}[x]$ and $\mathbb{R}[[x]]$ would actually be talking about different computations.

(c) A key fact about my solution is that my formula for each b_i involves all the b_j before it, for $j < i$. If you wanted to be very rigorous you could write a little proof by induction to check that every b_i is indeed well-defined, but I wasn't expecting that from you. More importantly, the proof does not require me to solve for an expression for each b_i just in terms of the a_i. There is in fact a lovely formula for the b_i in terms of the a_i alone, but it involves some concepts from combinatorics, namely integer partitions and multinomial coefficients, that you have likely not encountered.

I also didn't annotate my proof in part (b) to say which field axioms I was using when, nor did I expect you to. But if you're very assiduous, this is an exercise you could do, for $K[[x]]$ where K is any field. For example, how many times did I have to use the additive commutative law to do all the cancellations when proving the ith coefficient of ab is zero?

The assumption that multiplication in $\mathbb{R}[[x]]$ is commutative isn't really necessary for the proof, but I didn't want to make you go through all the calculations in the proof twice.