Queen Mary University of London

MTH4104 **Introduction to Algebra**

Week 5 tutorial questions **w/c 21 February 2022**

This sheet contains questions for you to work through in your tutorial, singly or in a group.

It's important to work through lots of questions for practice. Remember that mathematics is not a spectator sport! If you want more questions, look at the "Extra questions" sheets on QMPlus.

1 Define the two sets

$$O = \left\{ \frac{a}{b} : a, b \in \mathbb{Z}, \gcd(a,b) = 1, b \text{ is odd} \right\}$$
$$E = \left\{ \frac{a}{b} : a, b \in \mathbb{Z}, \gcd(a,b) = 1, b \text{ is even}, b \neq 0 \right\}.$$

In other words, a fraction in simplest form belongs to O if its denominator is odd and to E if its denominator is even.

Is O a ring? Is O a field? Is E a ring? Is E a field? In the definitions of ring and field, use the ordinary addition and multiplication operations for rational numbers.

2 (a) What is the smallest subset of $\mathbb{R}$ that is a field?

(b) What is the smallest subset of $\mathbb{R}$ containing $\sqrt{2}$ that is a field?

(c) What is the smallest subset of $\mathbb{R}$ containing $\sqrt{2}$ and $\sqrt{3}$ that is a field?

Justify your answers. For all three parts, use the usual definitions of addition and multiplication for $\mathbb{R}$.

3 Write down a complete proof of the right distributive law for multiplication in $\mathbb{C}$.

[That is, don't only write down the manipulation of equations in the middle; you should have some opening and closing text.]

4 Let R be the set of all functions from $\mathbb{Z}$ to $\mathbb{Z}$, with addition defined by addition of values,

$$(f+g)(x) = f(x) + g(x) \qquad \text{for all } x \in \mathbb{Z},$$

and multiplication defined as composition,

$$(f \cdot g)(x) = f(g(x)) \qquad \text{for all } x \in \mathbb{Z}.$$

Prove that R is *not* a ring when given these operations. In other words, find a ring axiom that is not satisfied, and give a counterexample.