Queen Mary University of London

MTH4104 **Introduction to Algebra**

Week 5 coursework **due 28 February 2022**

You are to write up a careful and professionally presented solution to the question below. This is to be submitted on QMPlus as a single PDF or JPEG file by 12:00 noon, Monday 28 February 2022.

Question to submit Let S be the set

$$S = \{a + b\mathrm{k} : a, b \in \mathbb{R}\},$$

where k is a formal symbol. Define addition and multiplication operations on S as follows: given elements $x = a + b\mathrm{k}$ and $y = c + d\mathrm{k}$ in S,

$$x + y := (a + c) + (b + d)\mathrm{k},$$
$$xy := (ac + bd) + (ad + bc)\mathrm{k}.$$

(a) Prove both identity laws for S. Include a short explanation (one sentence is fine) of how you know what the identity elements are. (5 marks)

(b) Prove that the multiplicative inverse law is false for S. [That is, don't just write down a counterexample; also prove that your counterexample is valid.] (5 marks)

Solution

(a) **Short model proof** The additive identity element is $0 + 0\mathrm{k}$, and the multiplicative identity element is $1 + 0\mathrm{k}$.

Let $a + b\mathrm{k}$ be any element of S. Then

$$(a + b\mathrm{k}) + (0 + 0\mathrm{k}) = (a + 0) + (b + 0)\mathrm{k} = a + b\mathrm{k}$$
$$(0 + 0\mathrm{k}) + (a + b\mathrm{k}) = (0 + a) + (0 + b)\mathrm{k} = a + b\mathrm{k}$$

which proves the additive identity law. Likewise,

$$(a + b\mathrm{k})(1 + 0\mathrm{k}) = (a \cdot 1 + b \cdot 0) + (a \cdot 0 + b \cdot 1)\mathrm{k} = a + b\mathrm{k}$$
$$(1 + 0\mathrm{k})(a + b\mathrm{k}) = (1a + 0b) + (1b + 0a)\mathrm{k} = a + b\mathrm{k},$$

proving the multiplicative inverse law.

We know that $0 + 0\mathrm{k}$ and $1 + 0\mathrm{k}$ are the identity elements because the above computations worked out correctly.

Commentary We didn't know to start out with that $0+0\mathrm{k}$ and $1+0\mathrm{k}$ would be the identity elements. In theory, you should work out what the additive identity element is by solving the equation

$$(a+b\mathrm{k})+(c+d\mathrm{k})=(a+b\mathrm{k})$$

for c and d, after using the definition of addition to expand the left hand side, and similarly for multiplication. However, based on the similarity between $\mathbb{C}$ and S, you might have just guessed (correctly) that $0+0\mathrm{k}$ and $1+0\mathrm{k}$ were the identity elements.

In the model proof I added and multiplied x with the identity element in both orders because that matches the way the axioms are written.

(b) **Short model proof** A counterexample to the multiplicative inverse law is $x=1+\mathrm{k}$. Assume $y=a+b\mathrm{k}\in S$ were the multiplicative inverse of x, for some real numbers x and y. We compute

$$xy=(1+\mathrm{k})(a+b\mathrm{k})=(a+b)+(a+b)\mathrm{k}.$$

By assumption this product equals $1+0\mathrm{k}$. But this is a contradiction, because it would require $a+b=1$ and $a+b=0$ to be true simultaneously.

Commentary How could you find this counterexample? You could assume that the multiplicative inverse law was true, try to solve for a formula for the inverse of an element $c+d\mathrm{k}$, and look for when something goes wrong with the formula. In this case, if $c+d\mathrm{k}$ has inverse $a+b\mathrm{k}$, then

$$(a+b\mathrm{k})(c+d\mathrm{k})=(ac+bd)+(ad+bc)\mathrm{k}=1+0\mathrm{k},$$

so $ac+bd=1$ and $ad+bc=0$. This is a system of two linear equations in the two real unknowns a and b (regarding c and d as known). You can solve it in whatever way you're used to, giving the solution

$$a=\frac{c}{c^2-d^2},\quad b=\frac{-d}{c^2-d^2}.$$

If $c^2-d^2\neq 0$ then in fact this does give a formula for the inverse of $c+d\mathrm{k}$. But it suggests that the elements $c+d\mathrm{k}$ with $c^2-d^2=0$, i.e. with $c=\pm d$, might not have an inverse. This is why I chose $c=d=1$ for the short model proof above. (In fact no element $c+d\mathrm{k}$ with $c=\pm d$ has an inverse.) Remember that, whatever counterexample you give, it must not be the additive identity element $0+0\mathrm{k}$ that we found in part (a).

Unlike part (a) we don't have to mention both parts of the inverse law here. We assumed $xy=yx=1$, but only had to use $xy=1$ in the argument, which is fine.