Queen Mary University of London

MTH4104 **Introduction to Algebra**

Week 3 tutorial questions **w/c 7 February 2022**

This sheet contains questions for you to work through in your tutorial, singly or in a group.

It's important to work through lots of questions for practice. Remember that mathematics is not a spectator sport! If you want more questions, look at the "Extra questions" sheets on QMPlus.

1 Write out addition and multiplication tables for $\mathbb{Z}_5$, like the ones for $\mathbb{Z}_4$ in the lecture notes.

That is, let $S := \{0,1,2,3,4\}$. For each pair of elements $a,b \in S$, your tables should give the integers $c,d \in S$ such that

$$[a]_5 + [b]_5 = [c]_5 \qquad \text{and} \qquad [a]_5[b]_5 = [d]_5.$$

2 Find all solutions $X \in \mathbb{Z}_{10}$ to the equation $X^2 = [3]_{10} \cdot X$.
[Notation: X^2 means $X \cdot X$.]

3 Let m be a positive integer. Prove that the sum of all elements of $\mathbb{Z}_m$ equals $[0]_m$ if and only if m is odd.

4 (a) Write out the multiplication table for $\mathbb{Z}_6$. How many times does $[0]_6$ appear in each row of the table?

(b) Based on your answer to part (a) (and other examples, if it helps), write down a formula for how many times $[0]_m$ appears in the $[a]_m$ row of the multiplication table for $\mathbb{Z}_m$, for any integer a and natural number m. Can you explain why your formula works?

5 This question compares a naïve way to take the "sum" and "product" of two sets of integers to the definitions that we actually use in modular arithmetic.

(a) Prove that, for any integer $m > 0$, if X and Y are congruence classes of $\equiv_m$, then the set

$$S = \{x + y : x \in X, y \in Y\}$$

is a congruence class of $\mathbb{Z}_m$, and in fact equals the sum $X + Y$ within $\mathbb{Z}_m$.

(b) Give an example of an integer $m > 0$ and two congruence classes X, Y of $\equiv_m$ such that the set

$$P = \{xy : x \in X, y \in Y\}$$

is *not* the product XY within $\mathbb{Z}_m$.

Write down a general statement about P is related to XY.