Queen Mary University of London

MTH4104 Introduction to Algebra

Week 2 tutorial questions — w/c 31 January 2022

This sheet contains questions for you to work through in your tutorial, singly or in a group.

It's important to work through lots of questions for practice. Remember that mathematics is not a spectator sport! If you want more questions, look at the "Extra questions" sheets on QMPlus.

Question 1 Let R be the relation on the set of positive integers such that R is true of two positive integers if and only if there is no multiple of π between them.

(a) Write down R in symbols as a subset of $(\mathbb{Z}_{>0})^2$. [Be careful with "between": have you covered all cases?]

(b) Prove that R is an equivalence relation. You may assume that π is irrational.

(c) Write down the equivalence class of 24 in R. You may use a calculator.

Solution (a) The hint is to remind you that, in terms of inequalities, there are two ways for a real number c to be between two other real numbers a and b: either $a < c < b$ or $b < c < a$. Therefore

$$R = \{(a,b) \in (\mathbb{Z}_{>0})^2 : \nexists k \in \mathbb{Z} \text{ such that } a < k\pi < b \text{ or } b < k\pi < a\}.$$

As I say in the notes, I don't mind if you use $\mathbb{N}$ instead of $\mathbb{Z}_{>0}$.

(b) We must prove reflexivity, symmetry, and transitivity.

Reflexivity. Let $a \in \mathbb{Z}_{>0}$. There can exist no real number c such that $a < c < a$ (whether or not c is a multiple of π). So $(a,a) \in R$. Therefore R is reflexive.

Symmetry. Informally, R is symmetric because the two endpoints in a "between" statement have symmetrical roles. Suppose $a, b \in \mathbb{Z}_{>0}$ satisfy $(a,b) \in R$. Then there is no $k \in \mathbb{Z}$ such that $a < k\pi < b$ or $b < k\pi < a$. But, swapping that round to $b < k\pi < a$ or $a < k\pi < b$, this is exactly the same condition as $(b,a) \in R$. So R is symmetric.

Transitivity. Let a,b,c be natural numbers such that $(a,b) \in R$ and $(b,c) \in R$. We wish to prove that $(a,c) \in R$. I will instead prove the contrapositive. [You don't have to do this, but a hint that the contrapositive, or proof by contradiction, might be helpful is that the statement inside the definition of R starts with a "there does not exist".]

So assume $(a,c) \notin R$. This means there exists an integer k such that $a < k\pi < c$ or $c < k\pi < a$. I'll start with the first case, $a < k\pi < c$. Because π is irrational, $k\pi$ is not an integer, so it doesn't equal b; therefore it is either less than or greater than b. But if $k\pi < b$, then $a < k\pi < b$ so $(a,b) \notin R$, whereas if $b < k\pi$, then $b < k\pi < c$ so $(b,c) \notin R$. That proves this first case of the contrapositive.

Finally, coming back to the other case $c < k\pi < a$, this is proved the same way as the previous case except with the roles of a and c reversed. [That is, I won't write the whole proof out, because I would literally do that just by copying and pasting and then switching as and cs. You are allowed to use wording like this, or "without loss of generality", in your own proofs too. Just make sure the two parts of the proof do match perfectly and you're not hiding anything.]

(c) The equivalence class of 24 consists of all integers b such that $(24,b) \in R$, i.e. such that there are no multiples of π between 24 and b. This is the set of integers between the two multiples of π closest to 24 on either side. Calculation shows these are $7\pi = 21.9911\ldots$ and $8\pi = 25.1327\ldots$, so

$$[24]_R = \{22,23,24,25\}.$$

Question 2 If R is a relation on a set X, then you need to state what X is to fully specify the relation (just like you need to state the domain and codomain to fully specify a function). This question is about why X is needed.

(a) Give an example of a set R of ordered pairs and two different sets X and Y such that R is a relation on X and also a relation on Y.

(b) If X and Y are different sets, is it possible for R to be an equivalence relation on X and also an equivalence relation on Y? Justify your answer.

Solution (a) To give a small example (but not the smallest I could!), $\{(1,1)\}$ is a relation on both the set $\{1\}$ and the set $\{1,2\}$.

(b) No, it's not possible. Here is a proof that, if R is an equivalence relation on X and also an equivalence relation on Y, then X and Y are equal.

Let x be an element of X. Since R is an equivalence relation on X, it is reflexive, which means that the pair (x,x) is an element of R. But since R is a relation on Y, it is a subset of Y^2, which means that $(x,x) \in R$ must also be an element of

$$Y^2 = \{(y_1,y_2) : y_1,y_2 \in Y\}.$$

Therefore x is an element of Y. This proves that X is a subset of Y.

Now, X and Y have symmetrical roles in the question, so the very same argument with X and Y swapped will also prove that Y is a subset of X. Together, these statements $X \subseteq Y$ and $Y \subseteq X$ imply $X = Y$, completing the proof.

Question 3 Let $X = \{2,4,6,8,10\}$ and let the partition $\{A_1, A_2, A_3\}$ of X be defined by

$$A_1 = \{2,8\}, \quad A_2 = \{4,6\}, \quad A_3 = \{10\}.$$

Write down the equivalence relation $R \subseteq X \times X$ determined by this partition.

Solution The equivalence relation determined by a partition relates two elements of X precisely when they lie in the *same* part of the partition. More formally: let $x, y \in X$, let x lie in part A_i and let y lie in part A_j for some indices i and j. Then $(x,y) \in R$ if and only if $i = j$. So for example $(2,8) \in R$ and $(6,4) \in R$ but certainly $(2,4) \notin R$. Thus

$$\begin{array}{llllll} R = \{ & (2,2), & & & (2,8), & \\ & & (4,4), & (4,6), & & \\ & & (6,4), & (6,6), & & \\ & (8,2), & & & (8,8), & \\ & & & & & (10,10) \ \}. \end{array}$$

I have written the answer like this with spaces in to visually suggest how R is a subset of X^2, whose elements could be written out in a 5×5 grid. You don't need to include the spaces in your solution, of course.

Question 4 If r and s are real numbers, $\max(r,s)$ is defined to be whichever of r and s is greater. For example, $\max(5,3) = 5$.

Let R be the following relation on the set $\mathbb{R}^2$. For two points $p, q \in \mathbb{R}^2$, let $p = (x_1, y_1)$ and $q = (x_2, y_2)$. Then $(p,q) \in R$ if and only if $\max(|x_1|,|y_1|) = \max(|x_2|,|y_2|)$. You may assume R is an equivalence relation [i.e. you need not write out a proof of this].

State the partition associated to R in words. Give the nicest, least "robotic" description you can.

Solution One new feature of this question is that the set called X in the notes is a set of pairs, namely $X = \mathbb{R}^2$. Therefore a relation on X is a set of pairs *of pairs* of real numbers! Thinking of elements of $\mathbb{R}^2$ as points in the plane might help prevent you from getting the two types of pairs confused.

Let $p = (x_1, y_1) \in \mathbb{R}^2$. To work out whether R is true or false we will be using the quantity $\max(|x_1|,|y_1|)$, so let us give this number a name, say $d = \max(|x_1|,|y_1|)$. Note that d is a non-negative number.

The equivalence class $[p]_R$ is the set of points $q = (x,y)$ such that $(p,q) \in R$, that is such that $\max(|x|,|y|) = d$. How can we describe these sets nicely? Pick a value (or a few values) of d and try sketching the set in the plane. For example, if $d = 2$, then $q \in [p]_R$ if and only if one of $|x|$ and $|y|$ is 2 and the other is not greater than 2. If $|x| = 2$ then either $x = 2$ or $x = -2$; similarly for y. Thus $[p]_R$ consists of two horizontal and two vertical line segments, through the points ± 2 on the x and y-axis, forming a square. Any other $d > 0$ will give a square in the same way, while for $d = 0$ we just get $[p]_R = \{(0,0)\}$.

The partition associated to R is the set of all these congruence classes. If you're willing to call a single point a square, then:

> The partition associated to R is the set of all squares with horizontal and vertical sides and centre $(0,0)$.

(If you're unwilling, you would have to tack on "together with the set $\{(0,0)\}$" or similar words.)

Question 5 The following is a false proposition; it is missing a necessary assumption.

> **Proposition.** Let X be any set and Y any subset of X. Then $\{Y, X \setminus Y\}$ is a partition of X.

Set the proposition right by including the necessary assumption, and then prove it.

Solution The necessary assumption is that Y must be a nonempty proper subset of X: it cannot be the empty set, and it cannot equal X. In these two cases, one of the parts of the putative partition will be an empty set.

To prove the amended proposition, we must check that when A is a nonempty proper subset of X, the set of sets $P = \{Y, X \setminus Y\}$ satisfies the three properties of a partition.

- We must show that none of the sets in P is the empty set. But Y is not empty by assumption, and $X \setminus Y$ is not empty because Y is a proper subset of X, so that at least one element of X is not in Y and is therefore in $X \setminus Y$.
- We must show that any two different sets in P are disjoint. But there is only one way to choose two different sets from P, namely Y and $X \setminus Y$. These are disjoint because no element of Y is contained in $X \setminus Y$, by definition of the set difference.
- Finally, we must show that the union of all the sets in P is the set X, that is that

  $$Y \cup (X \setminus Y) = X.$$

  Both Y and $X \setminus Y$ are subsets of X, establishing the $\subseteq$ inclusion. For the $\supseteq$ inclusion, given any element $x \in X$, we have two cases: either $x \in Y$, or $x \notin Y$. In the latter case $x \in X \setminus Y$, so in either case $x \in Y \cup (X \setminus Y)$. This completes the proof.