Remarks (a) Note that the conjugacy equation $h \circ f = g \circ h$, can be equivalently written as $f = h^{-1} \circ g \circ h$, or $g = h \circ f \circ h^{-1}$

i.e. $h(f(x)) = g(h(x)) \forall x \in X$

(b) Conjugacy can be viewed as a change of coordinates/variables.

So if we want to study the orbit of $x_0$ under $f$ in $X$, we can use $h$ to equivalently study the orbit of $y_0 = h(x_0)$ under $g$ in $Y$. 
(c) Another way of thinking of the conjugacy equation is that it is saying that the following diagram commutes:

\[
\begin{array}{ccc}
X & \xrightarrow{f} & X \\
\downarrow{h} & & \downarrow{h} \\
Y & \xrightarrow{g} & Y
\end{array}
\]

(d) Sometimes we may work with the case where \(X\) and \(Y\) are the same interval.

(e) If \(\phi: X \rightarrow Y\) is a topological conjugacy from \(f\) to \(g\), then we also say that \(f\) is topologically conjugate to \(g\).

**Lemma**

Topological conjugacy is an equivalence relation. [Where \(\sim\) means “is topologically conjugate to”]

Recall: An equivalence relation \(\sim\) must satisfy

- Reflexivity: \(a \sim a\) for all \(a\)
- Symmetry: \(a \sim b \Rightarrow b \sim a\) for all \(a, b\)
- Transitivity: \(a \sim b\) and \(b \sim c\) then \(a \sim c\)
Proof: Reflexivity: \( f: X \to X \) is clearly topologically conjugate to itself since \( h = \text{id}: X \to X \) satisfies

\[ h \circ f = f = f \circ h, \]
and this \( h \) is a homeomorphism.

Symmetry: Suppose \( f: X \to X \) is topologically conjugate to \( g: Y \to Y \), i.e., there exists a homeomorphism \( h: X \to Y \) satisfying \( h \circ f = g \circ h \). Since \( h \) is a homeomorphism, then so is \( h^{-1} \), and moreover from (*) we have

\[ h^{-1} \circ (h \circ f) \circ h^{-1} = h^{-1} \circ (g \circ h) \circ h^{-1} \]

thus

\[ h^{-1} \circ g = f \circ h^{-1} \]

Transitivity: Check this as an exercise. \( \Box \)
Lemma  If \( h: X \to Y \) is a topological conjugacy from \( f: X \to X \) to \( g: Y \to Y \), then \( h \) is also a topological conjugacy from \( f^n \) to \( g^n \) for all \( n \in \mathbb{N} \).

Proof  Let us write the conjugacy equation as \( f = h^{-1} \circ g \circ h \).

Then \( f^n = (h^{-1} \circ g \circ h) \circ (h^{-1} \circ g \circ h) \circ \ldots \circ (h^{-1} \circ g \circ h) \) \( n \) times, using that \( h \circ h^{-1} = \text{id} \) \((n-1)\) times

\[ = h^{-1} \circ g^n \circ h \]

i.e. \( h \circ f^n = g^n \circ h \)

Thus \( h \) is a top. conj. from \( f^n \) to \( g^n \).

\[ \square \]
We can use this lemma to show:

**Proposition**  Topological conjugacy preserves orbits, periodic points, and prime periods of orbits.

**Proof**  Let \( h : X \to Y \) be a topological conjugacy from \( f : X \to X \) to \( g : Y \to Y \). Let \( x_0 \in X \) and let \( y_0 = h(x_0) \in Y \).

Examine \( O^+(x_0) = \{ f^n(x_0) : n \geq 0 \} \) (the orbit of \( x_0 \) under \( f \)).

Then \( h(O^+(x_0)) = \{ h(f^n(x_0)) : n \geq 0 \} \)

\[ = \{ g^n(h(x_0)) : n \geq 0 \} \]

\[ = \{ g^n(y_0) : n \geq 0 \} \] (orbit of \( y_0 \) under \( g \))

\[ = O^+(y_0) \] (by the previous lemma)
So we have seen that the image of an orbit is itself an orbit.

Now let $x_0$ be a point of prime period $k$ under $f$.

This means that $f^k(x_0) = x_0$,

and $f^m(x_0) \neq x_0$ for $0 < m < k$.

Let $y_0 = h(x_0)$, and consider $g^k(y_0)$:

\[
g^k(y_0) = g^k(h(x_0)) = h(f^k(x_0)) = h(x_0) = y_0
\]

Moreover, for $0 < m < k$,

\[
g^m(y_0) = g^m(h(x_0)) = h(f^m(x_0)) \neq h(x_0) = y_0,
\]

so $y_0$ has prime period $k$. □