Queen Mary
University of London

Main Examination period 2022 – May/June – Semester B

MTH7000 / MTH7999: Postgraduate module

You should attempt ALL questions. Marks available are shown next to the questions.

In completing this assessment:

- **You may use books and notes.**
- **You may use calculators and computers, but you must show your working for any calculations you do.**
- **You may use the Internet as a resource, but not to ask for the solution to an exam question or to copy any solution you find.**
- **You must not seek or obtain help from anyone else.**

All work should be **handwritten** and should **include your student number**.

The exam is available for a period of **24 hours**. Upon accessing the exam, you will have **4 hours** in which to complete and submit this assessment.

When you have finished:

- scan your work, convert it to a **single PDF file**, and submit this file using the tool below the link to the exam;
- e-mail a copy to **maths@qmul.ac.uk** with your student number and the module code in the subject line;
- with your e-mail, include a photograph of the first page of your work together with either yourself or your student ID card.

Please try to upload your work well before the end of the submission window, in case you experience computer problems. **Only one attempt is allowed – once you have submitted your work, it is final**.

Examiners: F. Examiner, S. Examiner

This is optional. Here, one should put any special instructions (if any) relating to the exam. Otherwise, do not use this functionality.
For example: In this exam $0 \in \mathbb{N}$. Also, all expressions should be simplified as much as possible.

Question 1 [15 marks].

(a) Give a useful parametrisation of all solutions in positive integers x, y, z to the equation $x^2 + y^2 = z^2$. [**5**]

(b) Prove that the equation $x^4 + y^4 = z^2$ has no solutions in which x, y and z are positive integers. [**5**]

(c) Classify all quadruples (x, y, z, n) of positive integers such that $n \geq 3$ and the equation $x^n + y^n = z^n$ holds. Justify your answer. [**5**]

Question 2 [25 marks].

(a) Define what it means for a function to be **analytic** on some open subset of $\mathbb{C}$. [**5**]

Let s be a complex number such that $\Re(s) > 1$. Define $\zeta(s)$ to be the number:

$$\zeta(s) := \sum_{n=1}^{\infty} n^{-s}.$$

(In all cases, n^{-s} means $\exp(-s \log n)$, with $\log n \in \mathbb{R}$.)

(b) State how to define the analytic continuation of ζ to all of $\mathbb{C} \setminus \{1\}$. [**10**]

(c) Show that all non-real zeros of this analytic continuation have real part equal to $\frac{1}{2}$. [**10**]

Question 3 [20 marks].

(a) We consider packings in two dimensions.

(i) Prove that no packing of unit circles in the Euclidean plane has density exceeding $\frac{\pi}{2\sqrt{3}}$. [**5**]

(ii) What is the maximum possible density for packing ellipses with semi-minor axis 1 and semi-major axis a into Euclidean space? Justify your answer. [**5**]

(b) (i) Prove **without the assistance of a computer** that no packing of unit spheres in $\mathbb{R}^3$ has density exceeding $\frac{\pi}{3\sqrt{2}}$. [**5**]

(ii) Generalise your result to packing ellipsoids whose semi-axes are a, b and c. [**5**]

Question 4 [10 marks]. Prove that any even integer at least 8 can be written as the sum of two distinct (positive) primes.

Question 5 [30 marks]. State and prove the Classification of Finite Simple Groups. *[You must clearly establish the existence of all groups concerned, especially if they are defined by having involution centralisers of a certain type.]* [**30**]

End of Paper – An appendix of 2 pages follows.

A (hyper-)volume formula

The formula for the (hyper-)volume of an n-dimensional ball of radius r is

$$\frac{(\pi r^2)^{n/2}}{\Gamma(\frac{n}{2}+1)},$$

where, for $x > 0$, $\Gamma(x)$ is defined to be the real integral $\int_0^\infty t^{x-1}\mathrm{e}^{-t}\mathrm{d}t$.

Some trigonometric identities

$$\begin{aligned}
\sin^2 A + \cos^2 A &= 1\\
\tan^2 A + 1 &= \sec^2 A\\
1 + \cot^2 A &= \operatorname{cosec}^2 A\\
\sin(A \pm B) &= \sin A \cos B \pm \cos A \sin B\\
\cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B\\
\tan(A \pm B) &= \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}
\end{aligned}$$

Some derivatives

In the table below, a, b and c are constants.

$f(x)$	$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\,(f(x))$
ax^b	abx^{b-1}
$cf(x)$	$cf'(x)$
$f(x) \pm g(x)$	$f'(x) \pm g'(x)$
$f(x)g(x)$	$f'(x)g(x) + f(x)g'(x)$
$f(g(x))$	$g'(x)f'(g(x))$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\log x$	$\frac{1}{x}$

End of Appendix.