What about order-reversing diffeomorphisms? Can they have periodic points which are not fixed points?

\[ f(x) = -x \]

Here, 0 is the fixed point, and every other real number has prime period 2.
Proposition 1. If \( f : \mathbb{R} \to \mathbb{R} \) is an order-reversing diffeomorphism, then it has no points of prime period \( k \) with \( k \geq 2 \).

Proof. Since \( f \) is a diffeomorphism, so is \( f^2 \). Notice also that \( f^2 \) is order preserving because

\[
(f^2)'(x) = \frac{f'(f(x)) \cdot f'(x)}{<0 <0},
\]

and \( f' < 0 \) everywhere, so \( f'(f(x)) < 0 \) and \( f'(x) < 0 \), therefore \( (f^2)'(x) > 0 \). By the previous Proposition, \( f^2 \) does not have any periodic points of prime period \( > 1 \).
i.e., there does not exist \( p \in \mathbb{R} \) such that \((f^2)^m(p) = p\), \( m > 1 \).

i.e. \( f^{2^m}(p) = p \), \( m > 1 \)

So \( f \) has no points of prime period \( 2m \) for \( m > 1 \).

i.e. \( f \) has no prime period \( k \) points for even numbers \( k \).

To address the case where \( k \) is odd, we notice

\( f \) order-reversing diffeomorphism \( \implies f^k \) order-reversing diffeomorphism

But order-reversing diffeomorphisms have precisely one fixed point, so there exists a unique \( p \in \mathbb{R} \) with \( f(p) = p \), so also \( f^k(p) = p \).
Thus the unique fixed point for $f^k$ is just the fixed point for $f$;
that is, the only period-$k$ points for $f$ are its fixed points, so in particular there are no points of prime period $k$ for $f$. □

Summarising the situation for all diffeomorphisms $f : \mathbb{R} \to \mathbb{R}$:

| $f$ order-preserving | Fixed Points | Prime period 2 | Prime period $\geq 2$
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<tr>
<td>Arbitrary number</td>
<td>None</td>
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<td>f order-reversing</td>
<td>Exactly One</td>
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Fixed points and periodic points of continuous maps $f : \mathbb{R} \to \mathbb{R}$

Proposition  Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. If $f$ has an orbit of prime period 2 then it has a fixed point.

Proof  Let $\{a, b\}$ be a 2-cycle of $f$, with $a < b$.

Then $f(a) = b$, and $f(b) = a$.

Let $g(x) = f(x) - x$, so that a zero of $g$ is a fixed point of $f$.

Note that $f$ is continuous on $[a, b]$, since it is continuous on the whole of $\mathbb{R}$.
Also $g(a) = f(a) - a$
\[= b - a > 0,\]
and $g(b) = f(b) - b$
\[= a - b < 0\]

By the Intermediate Value Theorem there exists $c \in (a, b)$ such that $g(c) = 0$, i.e., such that $f(c) = c$.

So $f$ has a fixed point $c$, as required. 

**Remark** This is an example of "forcing", i.e., presence of a periodic orbit of some type/period forces the presence of another type/orbit.
Perhaps the most famous result of this type is "Period-3 implies chaos", which more precisely states:

**Theorem**  If $f: \mathbb{R} \rightarrow \mathbb{R}$ is continuous and has an orbit of prime period 3, then it has periodic orbits of all other prime periods $n$, $n \in \mathbb{N}$. 