Due Thursday 18th October. Attempt answers to all questions.

Hand in your script to me during the lecture.

Field equations and symmetries [20 marks]

Problem 1

Consider the Lagrangian density for the two real scalar fields $\phi_1$ and $\phi_2$ given by

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_1)(\partial^\mu \phi_1) + \frac{1}{2}(\partial_\mu \phi_2)(\partial^\mu \phi_2) - \frac{m_1^2}{2}(\phi_1 \phi_1) - \frac{m_2^2}{2}(\phi_2 \phi_2) - g(\phi_1 \phi_2)^2.$$ 

Here $g$ denotes a constant (called “coupling constant” as it couples the fields $\phi_1$ and $\phi_2$).

(i) Find the Euler-Lagrange equations for $\phi_1$ and $\phi_2$. \[3\]

Consider the infinitesimal transformation parameterised by the infinitesimal parameter $\epsilon$

$$\delta \phi_1 = \epsilon \phi_2, \quad \delta \phi_2 = -\epsilon \phi_1.$$

(ii) Find the variation of $\mathcal{L}$ under this transformation. Hint: remember to vary both $\phi$ and $\partial_\mu \phi$, for both fields! \[3\]

(iii) Find the most general set of values for the parameters $m_1$, $m_2$ and $g$ such that this transformation is a symmetry of the Lagrangian. Note that in this case we do not touch the coordinates, hence effectively the request of symmetry

$$|J|L(\varphi'(x'), \partial'_\mu \varphi'(x')) = L(\varphi(x), \partial_\mu \varphi(x))$$

becomes

$$L(\varphi', \partial_\mu \varphi') = L(\varphi, \partial_\mu \varphi'),$$

or, infinitesimally, $\delta L = 0$ ($|J|$ is the Jacobian of the transformation of the coordinates). \[3\]

(iv) In the next lecture we will show that the “current” $\delta J^\mu$ given by

$$\delta J^\mu := \sum_i \frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi_i)} \delta \varphi_i + \mathcal{L} \delta x^\mu$$

is conserved, which means that

$$\partial_\mu J^\mu = 0,$$
as a consequence of Noether’s theorem in field theory. The sum over $i$ is extended to all fields in the theory, hence in this case $\phi_1$ and $\phi_2$ (also, for the particular transformation given in part (i), we have $\delta x^\mu = 0$).

For the choice of parameters identified in part (iii), write down the expression of the Noether current $\delta J^\mu$ associated to this symmetry using the formula given above. [3]

(v) Check explicitly that $\partial_\mu J^\mu = 0$, upon using the equations of motion for the fields $\phi_1$ and $\phi_2$. [3]

(vi) [Optional question] Determine, with an explicit calculation, the form of the finite transformation corresponding to $\delta \phi_1, \delta \phi_2$ given above. What kind of transformation on $(\phi_1, \phi_2)$ is this?

*Hint:* write the infinitesimal transformation as a matrix acting on the column vector $
\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ and exponentiate it.

**Problem 2**

Consider the following Lagrangian density, describing a massive vector field,

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m^2}{2} A_\mu A^\mu.$$  

(i) Write down the equations of motion for $A_\mu$. [3]

(ii) Consider the transformation $A_\mu \rightarrow A_\mu + \partial_\mu \Omega$, where $\Omega$ is a function of the coordinates. How does the field strength $F_{\mu\nu}$ transform? Find also how the Lagrangian density $L$ transforms. [2]