PROBLEM 1

In class we derived

\[ [L^i, H] = i \varepsilon^{ijk} \alpha_j \beta_k = -[S^i, H] \]

so that \[ [J^i, H] = 0 \] with \[ \vec{J} = \vec{L} + \vec{S} \]. Then

\[ \Rightarrow [L^i, H] = [L^i, H] L^i + L^i [L^i, H] = \]

\[ = i \varepsilon^{ijk} \alpha_j (p^k L^i + L^i p^k) \neq 0 \]

\[ \Rightarrow [J^i, H] = J^i [J^i, H] + [J^i, H] J^i = 0 + 0 = 0 \]

\[ \Rightarrow \left[ p^i \Sigma^i \right] H = \frac{p^i}{|p|} \left[ \Sigma^i \right] H = \]

\[ = \frac{p^i}{|p|} (-2i) \varepsilon^{ijk} \alpha_j \beta_k = 0 \quad \text{since } \varepsilon^{ijk} \alpha_j \beta_k = 0 \]

\[ \text{(and symmetric, } \alpha_i \times \alpha_j = 0 \text{!)} \]
(vi) A general formula is
\[ T^{\nu} = \sum \frac{\partial}{\partial x^m} \left( \sqrt{-g} \phi^n \right) - \partial_x \eta^{\nu} \]
where the sum is over all fields. Thus we get
\[ T^{\nu} = (\phi^m)^+ (\partial^\nu \phi) + (\partial^\nu \phi^m) (\partial^\nu \phi) = \partial_x \eta^{\nu} \]
with \[ \eta = (\phi^m)^+ (\partial^\nu \phi) - m^2 \phi^m \chi^m \]

\[ \Sigma J^\nu = \sum \frac{\partial}{\partial x^m} \delta \phi^m + \partial \delta x^m \]

Here \[ \delta x^m = \alpha^m g \times \delta x \]
and from \[ 0 = \delta^\phi = \delta^\phi + \delta^\phi \delta x^g \]
\[ \Rightarrow \delta \phi = - (\partial^\nu \phi) \delta x^g \]

\[ \Sigma J^\nu = (\phi^m)^+ \delta^\phi + \delta^\phi (\partial^\nu \phi) + \partial \delta x^m = \]
\[ = - (\phi^m)^+ (\partial^\nu \phi) \delta x^g - (\partial^\nu \phi^m) (\partial^\nu \phi) \delta x^g + \partial \delta x^m \]
\[ = - \left[ (\phi^m)^+ (\partial^\nu \phi) + (\partial^\nu \phi^m) + m^2 \right] \delta x^g \]
\[ = - T^{\nu} \delta x^g \Rightarrow \delta J^\nu = - T^{\nu} \delta x^g \]

Substituting \[ \delta x^g \] we get
\[ \delta J^\mu = -T^\mu \delta \phi_0 \quad \delta \phi_0 \times \delta = + \alpha \delta \phi_0 \cdot \frac{1}{2} \left( x^5 T^\mu - x^5 T^\kappa \right) \]

\[ = \frac{1}{2} \alpha \delta \phi_0 \quad \eta_{\mu \kappa} \]

\[ \eta_{\mu \kappa} = x^5 T^\mu - x^5 T^\kappa \]

Note that it is compulsory to antisymmetrise \( \alpha \delta \phi_0 \) before we drop \( \alpha \delta \phi_0 \) to get the finite currents.

Hence the finite currents are (dropping also the \( \frac{1}{2} \) precisely \( \eta_{\mu \kappa} \)).

**Bonus Question**

Noether's Theorem guarantees us that \( \eta_{\mu \kappa} \) is still it is satisfactory to check this explicitly. Thus

\[ \eta_{\mu \kappa} \]

\[ \eta_{\mu \kappa} = \delta_{\mu} (x^5 T^\kappa - x^5 T^\kappa) = \]

\[ = \delta_{\mu} T^\kappa \]

\[ = 0 \quad \text{by translational symmetry} \]

\[ = \alpha \delta \phi_0 \quad \text{as well} \]

\[ = T^0 - T^0 = 0 \quad \text{since with our expression for } T^\mu \text{ we can see that } T^0 = T^0. \]

\[ \text{for Klein-Gordon} \]