PROBLEM 1

(i) We have \( \partial \mu e^{-\text{i}px} = \text{i} p \mu e^{-\text{i}px} \),
\[
\partial e = \partial \mu e^{-\text{i}px} = (\text{i} p \mu) e^{-\text{i}px} = -p^2 e^{-\text{i}px}
\]
\[
\Rightarrow \quad (\Box + m^2) e^{-\text{i}px} = (\text{p}^2 + m^2) e^{-\text{i}px} = 0
\]
\[
\sqrt{-\text{p}^2 + \text{p}^2 + m^2} = -\text{p}^2 e^{\text{E}^2(\text{p}^2)}
\]
\[
\phi_0 = E(p^2), \quad \text{Thus}
\]
\[
(\Box + m^2) \phi(x) = \int d^3p \, N(p^2) \left[ a(p^2) (\Box + m^2) e^{-\text{i}px} + b(p^2) (\Box + m^2) e^{-\text{i}px} \right]
\]
\[
= 0 \quad \text{if} \quad \phi_0 = E(p^2).
\]

We use a different letter, \( b \), for the coefficient of \( e^{-\text{i}px} \) since \( \phi \) is complex. If \( \phi \) were real, the coefficient of \( e^{-\text{i}px} \) should have been \( a^+ \).

(ii) \( \phi(x) = \int d^3p \, N(p^2) \left[ a(p^2) e^{-\text{i}px} + b^+(p^2) e^{-\text{i}px} \right] \)
\[
\phi^+(y) = \int d^3q \, N(q^2) \left[ a^+(q^2) e^{\text{i}qy} + b(q^2) e^{\text{i}qy} \right]
\]

with \( N(p^2) = \frac{1}{(2\pi)^3(2E(p^2))} \) \( \Rightarrow \)
\[
[ \phi(x), \phi^+(y) ] = \int \frac{d^3p}{(2\pi)^3(2E(p^2))} \frac{d^3q}{(2\pi)^3(2E(q^2))} \cdot
\]
\[
\left\{ e^{\text{i}(p \cdot x - q \cdot y)} \left[ a(p^2), a^+(q^2) \right] + e^{\text{i}(p \cdot x + q \cdot y)} \left[ b^+(p^2), b(q^2) \right] +
\right.
\]
\[
+ e^{\text{i}(p \cdot x - q \cdot y)} \left[ b(p^2), a^+(q^2) \right] + e^{\text{i}(p \cdot x + q \cdot y)} \left[ a(p^2), b(q^2) \right]
\]
\[
\left. \right|_{=0} = 1 \right)
Using \([a(p), a^+(q)] = [\sigma^R(p), \sigma^L(q)] = (2\pi)^3 \delta^3(p-q) \) we get

\[
[\phi(x), \phi^+(y)] = \int \frac{d^3p}{(2\pi)^3(2E(p))} \left[ e^{-i\vec{p} \cdot (x-y)} + e^{i\vec{p} \cdot (x-y)} \right]
\]

where everywhere \(p_0 = E(p)\)

But we had defined \((i\Delta) (x) = \pm \int \frac{d^3p}{(2\pi)^3(2E(p))} e^{-i\vec{p} \cdot (x-y)} + iE(p)(x-y)\)

so clearly, \(\int \frac{d^3p}{(2\pi)^3(2E(p))} e^{-i\vec{p} \cdot (x-y)} + iE(p)(x-y)\) equals \(i\Delta(x-y)\)

For the second term, just observe that

\[
\int \frac{d^3p}{(2\pi)^3(2E(p))} e^{i\vec{p} \cdot (x-y)} = \int \frac{d^3p}{(2\pi)^3(2E(p))} e^{iE(p)(x-y) - i\vec{E}(p)(x-y)}
\]

\[
= \int \frac{d^3p}{(2\pi)^3(2E(p))} e^{i\vec{E}(p)(x-y) + i\vec{p}(x-y)}
\]

after changing integration variable

\(\vec{p} \rightarrow \vec{p}' = -\vec{p}\) and using that \(d^3p = d^3p'\), \(E(-\vec{p}) = E(\vec{p})\).

Thus \(- \int \frac{d^3p}{(2\pi)^3(2E(p))} e^{i\vec{p} \cdot (x-y)} \bigg|_{p_0 = E(p)} = -i\Delta^{(-)}(x-y)\)

and

\[
[\phi(x), \phi^+(y)] = i \left( \Delta^{(4)}(x-y) + \Delta^{(-)}(x-y) \right) = i\Delta(x-y)
\]

with \(\Delta(x) = (\Delta^{(4)} + \Delta^{(-)})(x)\).
\textbf{Problem 2}

\[(i) \quad \langle +, \vec{\kappa} | \phi(x) | 0 \rangle = \langle 0 | a(\vec{\kappa}) \phi(x) | 0 \rangle =\]

\[
= \int \frac{d^3p}{(2\pi)^3(2E(p))} \left[ \begin{array}{l}
- iE(p)t + i\vec{p} \cdot \vec{x} \\
+ \frac{1}{2} iE(p)t - i\vec{p} \cdot \vec{x}
\end{array} \right] \left[ \begin{array}{c}
\langle 0 | a(\vec{\kappa}) \phi(p) | 0 \rangle \\
\langle 0 | a(\vec{\kappa}) \phi(p) | 0 \rangle + \langle 0 | a(\vec{\kappa}) \phi(p) | 0 \rangle
\end{array} \right] = 0
\]

\[
= \langle 0 | [a(\vec{\kappa}), \phi(p)] | 0 \rangle = 0
\]

\[
\langle -, \vec{\kappa} | \phi(x) | 0 \rangle = \langle 0 | b(\vec{\kappa}) \phi(x) | 0 \rangle =
\]

\[
= 0 + \int \frac{d^3p}{(2\pi)^3(2E(p))} \left[ \begin{array}{l}
- iE(p)t - i\vec{p} \cdot \vec{x} \\
+ \frac{1}{2} iE(p)t - i\vec{p} \cdot \vec{x}
\end{array} \right] \left[ \begin{array}{c}
\langle 0 | b(\vec{\kappa}) \phi(p) | 0 \rangle \\
\langle 0 | b(\vec{\kappa}) \phi(p) | 0 \rangle + \langle 0 | b(\vec{\kappa}) \phi(p) | 0 \rangle
\end{array} \right] = 0
\]

\[
= \frac{\lambda^4 E(\vec{\kappa}) E(p)}{(2\pi)^3(2E(p))^2} \langle 0 | [b(\vec{\kappa}), \phi(p)] | 0 \rangle = 0
\]

\[
= 0
\]

\[
\text{The first two calculations thus give}
\]

\[
\langle +, \vec{\kappa} | \phi(x) | 0 \rangle = 0
\]

\[
\langle -, \vec{\kappa} | \phi(x) | 0 \rangle = \frac{i\hbar \cdot x}{\hbar} \quad (\text{with } \hbar = \hbar_c = E(\vec{\kappa}))
\]
Next

\[ \langle +, \hbar | \phi(x) | 0 \rangle = \int \frac{d^3p}{(2\pi)^3} e^{\frac{iE(p)t - ix \cdot p}{\hbar}} \langle 0 | a(\hbar^2) a^\dagger(p) | 0 \rangle + \]

\[ + \frac{\hbar k}{\hbar} x \]

\[ = 0 \text{ (with } k_0 = E(\hbar^2)) \]

and finally

\[ \langle -, \hbar | \phi(x) | 0 \rangle = \langle 0 | b(\hbar^2) b^\dagger(x) | 0 \rangle = 0 \text{ as it is impossible to form non-vanishing commutators} \]

Summarizing, for the last 2 cases

\[ \langle +, \hbar | \phi(x) | 0 \rangle = e^{\frac{i E(p) t - i x \cdot p}{\hbar}} \text{ (with } k_0 = E(\hbar^2)) \]

\[ \langle -, \hbar | \phi(x) | 0 \rangle = 0 \]
\( \langle 0 | a(k') a^+(p') | 0 \rangle = \langle 0 | [a(k'), b^+(p')] | 0 \rangle = 0 \)

\[
\langle 0 | a(k_1^\uparrow) a(k_2^\uparrow) a^+(p_1^\uparrow) a^+(p_2^\uparrow) | 0 \rangle = \\
= \langle 0 | a(k_1^\uparrow) [a(k_2^\uparrow), a^+(p_1^\uparrow)] a^+(p_2^\uparrow) | 0 \rangle + \langle 0 | a(k_1^\uparrow) a(p_1^\uparrow) a(k_2^\uparrow) a^+(p_2^\uparrow) | 0 \rangle = \\
= \langle 0 | a(k_1^\uparrow) a(k_2^\uparrow), a^+(p_1^\uparrow)] a^+(p_2^\uparrow) | 0 \rangle + \\
+ \langle 0 | a(k_1^\uparrow) a(p_1^\uparrow) [a(k_2^\uparrow), a^+(p_2^\uparrow)] | 0 \rangle = \\
= (2\pi)^3 (2E(k_2)) \delta^{(3)}(p_1^\uparrow - k_2^\uparrow) \langle 0 | a(k_1^\uparrow) a^+(p_1^\uparrow) | 0 \rangle + \\
+ (2\pi)^3 (2E(k_1)) \delta^{(3)}(p_2^\uparrow - k_2^\uparrow) \langle 0 | a(k_1^\uparrow) a^+(p_1^\uparrow) | 0 \rangle
\]

Next we notice that

\[
\langle 0 | a(k_1^\uparrow) a(p_1^\uparrow) | 0 \rangle = \langle 0 | [a(k_1^\uparrow), a^+(p_1^\uparrow)] | 0 \rangle = (2\pi)^3 (2E(k_1^\uparrow)) \delta^{(3)}(p_1^\uparrow - k_1^\uparrow)
\]

(3)

and similarly for \( \langle 0 | a(k_1^\uparrow) a(p_1^\uparrow) | 0 \rangle \).

\[
\langle 0 | a(k_1^\uparrow) a(k_2^\uparrow) a^+(p_1^\uparrow) a^+(p_2^\upright) | 0 \rangle = (2\pi)^6 (2E(k_1^\uparrow))(2E(k_2^\upright)) \\
= \delta^{(3)}(p_1^\uparrow - k_1^\uparrow) \delta^{(3)}(p_1^\uparrow - k_2^\upright) + \delta^{(3)}(p_2^\upright - k_2^\upright) \delta^{(3)}(p_2^\upright - k_1^\uparrow)
\]

which we could pictorially represent as

\[
\begin{array}{c}
\vec{p}_1 \rightarrow \vec{k}_1 \\
\vec{p}_2 \rightarrow \vec{k}_2
\end{array} + \\
\begin{array}{c}
\vec{p}_2 \rightarrow \vec{k}_1 \\
\vec{p}_1 \rightarrow \vec{k}_2
\end{array}
\]

\[
\begin{array}{c}
\vec{k}_1 \end{array} \\
\begin{array}{c}
\vec{k}_2
\end{array}
\]
The last case is simpler:

\[ \langle 0 | b(k_1) a(k_2) a^+(p_1) b^+(p_2) | 0 \rangle = \]

since the a's and the b's commute

\[ = \langle 0 | b(k_1) b^+(p_2) a(k_2) a^+(p_1) | 0 \rangle = \]

\[ = \langle 0 | [b(k_1), b^+(p_2)] [a(k_2), a^+(p_1)] | 0 \rangle = \]

\[ = (2\pi) b^6 (2\pi(k_1)) (2\pi(k_2)) S^{(3)}(p_1 - k_2) S^{(3)}(p_2 - k_1) . \]