Problem 1

(a) \( \phi^+ e^{-i\phi} \). Under phase transformations

\[ L \rightarrow e^{-i\phi'} e^{i\phi} (\partial \mu \phi^+)(x^\nu \phi) = L \]

The Noether current is

\[ \delta J^\mu = \frac{\partial L}{\partial (\partial_\nu \phi)} \delta \phi + \frac{\partial L}{\partial (\partial_\nu \phi^+)} \delta \phi^+ \]

\[ \delta \phi = i \partial \phi, \quad \delta \phi^+ = -i \partial \phi^+ \]

\[ \delta J^\mu = i \partial \phi^+ (\partial_\nu \phi) - i \partial \phi (\partial_\mu \phi^+) \]

The finite current is (keeping an \( i \) to make it hermitean)

\[ J^\mu = i (\phi^+ \partial_\mu \phi - \partial_\mu \phi^+ \phi) \]

(b) \[ S = \int d^4x (\partial_\mu \phi)(\partial^\mu \phi)(x) \rightarrow \]

\[ S' = \int d^4x'^4 (\partial_\mu ^\prime \phi(x'))(\partial^\mu ^\prime \phi(x')) \text{. Next we use} \]

\[ d^4x = e^4 d^4x', \quad \phi(x') = e \phi(x), \quad \partial_\mu ^\prime \equiv \frac{\partial}{\partial x'^\mu} = e \partial_\mu \]

\[ S' = \exp(-4a + 2a + 2a) \int d^4x (\partial_\mu \phi)(\partial^\mu \phi) = S \]
Thus \( S \) is invariant under dilatations.

\[
(c) \quad \phi(x') = e^a \phi(x), \quad \text{or} \quad \phi(e^a x) = e^a \phi(x),
\]

thus \( \phi(x) = e^{-a} \phi(e^a x) \) expanding for small \( a \)

we get

\[
\phi'(x) \sim (1 + a) \phi(x + a x) \sim (1 + a) \left[ \phi + a x \mu \partial^\mu \phi \right]
\]

\[
\sim \phi + a (\phi + x \mu \partial^\mu \phi) \quad \Rightarrow
\]

\[
\phi'(x) = \phi + a (1 + x \mu \partial^\mu) \phi \quad \text{and}
\]

\[
\delta \phi(x) = \phi'(x) - \phi(x) = a (1 + x \cdot \partial) \phi
\]

Similarly \( \delta \phi^+ = a (1 + x \cdot \partial) \phi^+ \) since \( a \in \mathbb{R} \).

Then we have

\[
\delta J^\mu = \frac{\Theta^\mu}{\Theta(\partial \mu \phi)} \delta \phi + \frac{\Theta^\mu}{\Theta(2 \mu \phi^+)} \delta \phi^+ + \Theta^\mu x\partial^\mu
\]

where \( x' = e^{-a} x \sim 1 - a x \Rightarrow \delta x^\mu = -a x^\mu \).
and using \( \frac{\partial \phi}{\partial (\partial \phi)} = \phi \phi^+ \)

\[ S J^\mu = \frac{1}{a^2} \left\{ (\partial \phi^+)(1+x\cdot\partial) \phi + (1+x\cdot\partial) \phi^+ (\partial \phi) - x^\mu \partial \right\} \]

and the finite current is

\[ J^\mu = (\partial \phi^+)(1+x\cdot\partial) \phi + (1+x\cdot\partial) \phi^+ \partial \phi - x^\mu \partial \]

Check current conservation on the solutions to the equations of motion \( \Box \phi = 0 \), \( \Box \phi^+ = 0 \):

\[ \partial_\mu J^\mu = 2 (\partial \phi^+)(\partial \phi) + (\partial \phi^+)(\partial \phi) - 4 (\partial \phi^+ \partial \phi) - x^\mu \partial \]

\[ \partial_\mu x^\mu = 0 \]

(d) Again it is very convenient to work at the level of the action. For the mass term we have

\[ S_m \rightarrow -m^2 \int d^4 x' \phi^+(x') \phi(x') = -m^2 e^{- \frac{1}{4} \lambda \int d^4 x' (\phi^+(x') \phi(x'))^2} \]

\[ S_m \rightarrow e^{- \frac{1}{4} \lambda \int d^4 x' (\phi^+(x') \phi(x'))^2} \]

\[ \frac{1}{4} \lambda \int d^4 x' (\phi^+(x') \phi(x'))^2 \]

\[ \rightarrow - \frac{4}{4} \lambda \int d^4 x (\phi^+ \phi)^2 \]

Then

\[ S_\lambda \rightarrow -\lambda \int d^4 x' \left( (\phi^+(x') \phi(x'))^2 \right) = - \lambda e^{- \frac{4}{4} \lambda \int d^4 x (\phi^+ \phi)^2} \]

\[ = \int d^4 x (\phi^+ \phi)^2 \]
Here \( S_1 \rightarrow S_1 \) under a dilatation. Hence for invariance we need \( m = 0 \), and \( \lambda \) arbitrary.

The dimensions: the key insight is that

\( S \) is ADIMENSIONAL \( \Rightarrow [S] = 0 \)

- \( [d^4 x] = -4 \) (in energy units).

What is the dimension of \( \phi \)? From the kinetic term \( \int d^4 x \left( \partial \phi \right) \left( \partial \phi \right) \) we get, imposing that the action is dimensionless,

\[-4 + 1 + 1 + 2[\phi] = 0 \Rightarrow [\phi] = 1\]

(since \([\partial] = +1\) \( \Rightarrow \)

\[\int d^4 x \phi \phi^* \] \( = -4 + 2 = -2 \) \( \Rightarrow [m^2] = +2 \)

\( \Rightarrow [m] = 1 \) has the dimension of a mass! \[ [m] = 1 \]

Next for \( S_1 \) :
\[
[\lambda] + [d^4x] + [\phi^2\phi^2] = 0 
\Rightarrow \\
[\lambda] = +4 - 4 = 0 
\Rightarrow \\
\lambda \text{ is dimensionless}
\]
\[ \Pi = \frac{\Theta \phi}{\Theta \phi} = \phi \]

\[ \mathcal{H}_0(\pi, \phi) = \pi (\Theta \phi) - L_0 = (\Theta \phi Y \Theta \phi) - L_0 = \]

\[ = \pi^2 - \frac{1}{2} \pi^2 + \frac{1}{2} (\vec{\phi} \cdot (\vec{\phi}) + \frac{1}{2} m^2 \phi^2 \]

\[ \Rightarrow \]

\[ \mathcal{H}_0(\pi, \phi) = \frac{1}{2} \pi^2 + \frac{1}{2} (\vec{\phi} \cdot (\vec{\phi}) + \frac{1}{2} m^2 \phi^2 \]

Then \[ H_0 = \int d^3 x \; \mathcal{H}_0 \quad \text{and} \]

\[ i^*(\Theta \phi)(x, t) = \left[ \phi(x, t), H_0 \right] = \int d^3 y \left[ \phi(x, t), \frac{i}{2} \pi(y, t) \right] \]

(\phi commutes with itself and with \( \vec{\phi} \) thus we can drop these terms in \( H_0 \) containing \( \phi \) and \( \vec{\phi} \))

\[ = \frac{2}{2} \int d^3 y \left[ \phi(x, t), \pi(y, t) \right] = \]

\[ \left[ i \delta^{(3)}(x-y) \right] \]

\[ = i \int d^3 y \; \delta^{(3)}(x-y) \; \pi(y, t) = i \pi(x, t) \quad \Rightarrow \]

\[ (\Theta \phi) = \pi \]

(20)
(c) The Dyson expansion for the S-matrix reads

\[ S = 1 + \sum_{n=1}^{\infty} \frac{(-i)^{n}}{n!} \int dt_{1} \ldots dt_{n} \ T(H_{1}(t_{1}) \ldots H_{n}(t_{n})) \]

where \( H_{ij} \) is the interaction Hamiltonian, or

\[ S = 1 + \sum_{n=1}^{\infty} \frac{(-i)^{n}}{n!} \int dx_{1} \ldots dx_{n} \ T(H_{1}(x_{1}) \ldots H_{n}(x_{n})) \]

In an case \( H_{ij} = - \frac{\partial^{2}}{\partial x^{2}} + \frac{1}{4} \phi \phi^{\dagger} \Rightarrow \]

\[ S = 1 + \sum_{n=1}^{\infty} \frac{(-i/4)^{n}}{n!} \int dx_{1} \ldots dx_{n} \ T(\phi^{\dagger}(x_{1}) \ldots \phi^{\dagger}(x_{n})) \]

(d) To the first order (in \( \lambda \)) we have

\[ S_{fi}^{(1)} = \langle 0 | e(p_{3}) a(p_{4}) (-i \lambda) \int dx : \phi(x) : e(p_{1}) a^{\dagger}(p_{2}) | 0 \rangle \]

out of \( \phi^{4} \) we need the term with

\( 2 \phi^{4} \) and with \( 2 \phi^{+} \phi^{-} \), this only will survive.

In the product \( \phi^{4} = (\phi^{+} \phi^{-})(\phi^{+} \phi^{-})(\phi^{+} \phi^{-})(\phi^{+} \phi^{-}) = \]

\( = \binom{4}{2} \phi^{+} \phi^{-} \phi^{+} \phi^{-} + \) terms which do not have \( 2 \phi^{+} \) and \( 2 \phi^{-} \).