The Dirac Lagrangian [20 marks]

Problem 1 – quick questions

Prove the following statements:

(i) \( p_\mu p_\nu \gamma^\mu \gamma^\nu = m^2 \) where \( p \) is the momentum of a particle of mass \( m \). \[2\]

(ii) \( p_\mu p_\nu \sigma^{\mu\nu} = 0 \), where \( \sigma^{\mu\nu} := (i/2)[\gamma^\mu, \gamma^\nu] \). \[2\]

(iii) \( (p_\mu \gamma^\mu - m)(p_\nu \gamma^\nu + m) = p^2 - m^2 \) (and hence = 0 is \( p \) is the momentum of a particle of mass \( m \)). \[2\]

Problem 2

(i) The Dirac Lagrangian density \( \mathcal{L} \) is given by \( \mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi \), where \( \bar{\psi} := \psi^\dagger \gamma^0 \).
Write down the momentum densities \( \pi_\psi \) and \( \pi_{\bar{\psi}} \) conjugate to \( \psi \) and \( \bar{\psi} \), respectively. Show that the Hamiltonian density is \( \mathcal{H} = \bar{\psi}(-i\vec{\gamma} \cdot \vec{\nabla} + m)\psi \). \[4\]

(ii) Find the equations of motion for the field \( \psi \) deriving from the Dirac Lagrangian in part (i). \[2\]

(iii) The solution to the equations of motion can be written as:
\[
\psi(x) = \int d^3p N(p) \sum_{r=1}^2 \left[ a_r(p)u_r(p)e^{-ip \cdot x} + b_r^\dagger(p)v_r(p)e^{ip \cdot x} \right],
\]
where in the above \( p_0 = E(p) \) and \( N(p) \) is a normalisation factor. Show that in order to satisfy the equations of motion, \( u_r(p) \) and \( v_r(p) \) must satisfy the equations
\[
(p_\mu \gamma^\mu - m)u_r(p) = 0 \quad \text{and} \quad (p_\mu \gamma^\mu + m)v_r(p) = 0 .
\][4]
Problem 3

The momentum of the Dirac field is given by $P^i = \int d^3x \ T^{0i}$, where the expression for the energy-momentum tensor $T^{\mu\nu}$ for the Dirac field is

$$T^{\mu\nu} = i \bar{\psi} \gamma^\mu \partial^\nu \psi - \eta^{\mu\nu} \mathcal{L},$$

with $\mathcal{L}$ being the Dirac Lagrangian density. Compute the expression for $P^i$ in terms of the creation and annihilation operators. \[4\]

In this problem you may use the normalisation $N(p) = 1/[(2\pi)^3 \sqrt{2E(p)}]$, which is consistent with the following relations that you may also use without proof:

$$u_r^\dagger(p)u_s(p) = v_r^\dagger(p)v_s(p) = \delta_{rs}(2E(p)),$$

$$u_r^\dagger(p)v_s(-p) = v_r^\dagger(p)u_s(-p) = 0.$$

You will also have to use that

$$\int d^3x \ e^{\pm ik \cdot x} = (2\pi^3)\delta^{(3)}(k).$$

Hope these hints will help you solve the problem more quickly! Your answer should be

$$P^i = \int \frac{d^3p}{(2\pi)^3} \ p^i \ \sum_r \left[ a_r^\dagger(p)a_r(p) - b_r(p)b_r^\dagger(p) \right].$$

The minus sign between the two terms is crucial and is related to the fact that the particles we are talking about are fermions! Make sure you reproduce it correctly...