Due Thursday 15th November. Attempt answers to all questions. Hand in your script to me during the lecture.

On the symmetries of the Dirac and Klein-Gordon equations [20+4 marks]

Problem 1 [6 marks]

In this problem we discuss operators which may or may not commute with the Dirac Hamiltonian $H = \alpha \cdot \hat{p} + \beta m$. We will have to use, among others, the results proved in class:

$[L^i, H] = -[S^i, H] = i \epsilon^{ijk} \alpha^j \hat{p}^k,$

so that $[J^i, H] = 0$, with $J^i = L^i + S^i$ and $S^i = \Sigma^i/2$, with

$\Sigma^i = \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}.$

Using also $[A, BC] = [A, B]C + B[A, C]$ compute the following commutators:

- $[L^2, H]$ [2]
- $[J^2, H]$ [2]
- $[h, H]$ where

$\quad h = \Sigma \cdot \hat{p} \\ |\hat{p}|$

is the so-called helicity operator. [2]

Problem 2 [14 marks]

Consider the complex Klein-Gordon theory Lagrangian density,

$L = (\partial^\mu \phi)^\dagger (\partial_\mu \phi) - m^2 \phi^\dagger \phi$.

(i) Use the formula for the energy momentum tensor $T^{\mu\nu}$ derived in class for a generic Lagrangian to compute $T^{\mu\nu}$ for the complex Klein-Gordon Lagrangian. [4]

(ii) Use Noether’s theorem to prove that there are six conserved currents, associated with Lorentz transformations, usually called $M^{\mu\nu\rho}$. Show that their expression is

$M^{\mu\nu\rho} = x^\rho T^{\mu\sigma} - x^\sigma T^{\nu\rho},$
with $M^{\mu;\rho\sigma} = -M^{\mu;\sigma\rho}$ (note that the indices $\rho$ and $\sigma$ label the different charges, while $\mu$ is the index of the current). [6]

Hints:

- Recall that scalar fields are invariant under Lorentz transformations, i.e.
  \[ \phi'(x') = \phi(x), \quad \text{hence} \quad \delta_T \phi = 0. \]
  In order to find $\delta \phi$, you may then use that $\delta_T \phi = \delta \phi + (\partial^\mu \phi) \delta x_\mu$.
- Remember to include in the sum the contributions from both $\phi$ and $\phi^\dagger$.
- Recall that for a Lorentz transformation $\delta x^\rho = \alpha^\rho_\sigma x^\sigma$ with $\alpha^\rho_\sigma = -\alpha^\sigma_\rho$.
- Then recall that whenever a tensor without manifest symmetry properties, say $B^{\rho\sigma}$, contracts to $\alpha^\rho_\sigma$, you should rewrite it by dropping its symmetric part, which contracts to zero: in formulae, $B^{\rho\sigma} \alpha^\rho_\sigma = (1/2) \alpha^\rho_\sigma (B^{\rho\sigma} - B^{\sigma\rho})$ since $\alpha^\rho_\sigma (B^{\rho\sigma} + B^{\sigma\rho})$ is identically zero.

(iii) Check that
\[ \partial_\mu M^{\mu;\rho\sigma} = 0, \]
by computing explicitly the derivative (and also using that $T$ is conserved). [4]