Example: Let \( f(x) = ax \) where \( a \in \mathbb{R} \).

Note that 0 is a fixed point of \( f \) (irrespective of the value of \( a \)).

Suppose \( |a| < 1 \)

Claim: 0 is an attracting fixed point

Proof: Choose any \( \delta > 0 \), let \( x_0 \in I = (-\delta, \delta) \)

Then \( \lim_{n \to \infty} f^n(x_0) = \lim_{n \to \infty} a^n x_0 = 0 \) since \( |a| < 1 \)

\( f^2(x) = f(f(x)) = f(ax) = a(ax) = a^2x \), etc.

And in general \( f^n(x) = a^n x \)
Suppose $|a| > 1$.

**Claim:** $0$ is a repelling fixed point.

**Proof.** Let $x_0 \in I = (-\delta, \delta)$. Then

$$|f^n(x_0)| = |a^n x_0| = |a|^n |x_0| > \delta$$

for $n$ sufficiently large $\Box$

**Exercise:** Let $f(x) = ax + b$.

where $a \neq 0$, $a \neq \pm 1$.

Find the fixed point for $f$ and determine whether it is attracting or repelling.

**Example:** Let $f(x) = x^2$. Determine whether the fixed points at $p = 0$ and $p = 1$ are attracting or repelling.

For $p = 0$ we showed that for $|x| < 1$ then $x \in W^s(0)$. Thus $0$ is attracting.

For $p = 1$, we can use an orbit web to make the guess that it is repelling.
To make rigorous, we could choose $I = (1-\delta, 1+\delta)$ for some $\delta > 0$, for example $\delta = \frac{1}{2}$, and show that if $x_0 \in I \setminus \{1\} = (1-\delta, 1) \cup (1, 1+\delta)$ then $f^n(x_0) \not\in I$ for some $N \in \mathbb{N}$.

It would be useful to have a convenient condition to check if/when a fixed point is attracting or repelling.

**Defn:** A function $f: \mathbb{R} \to \mathbb{R}$ is said to be $C^1$ if $f$ is differentiable and its derivative $f'$ is continuous.
Theorem. If \( f : \mathbb{R} \to \mathbb{R} \) is \( C^1 \), and \( p \) is a fixed point of \( f \), then
\( p \) is attracting if \( |f'(p)| < 1 \),
and \( p \) is repelling if \( |f'(p)| > 1 \).

Idea of proof:

Before proving this theorem, recall the following

Mean Value Theorem (MVT)

If \( f : [a, b] \to \mathbb{R} \) is \( C^1 \) then there exists \( \xi \in (a, b) \) such that
\[
f'(<\xi) = \frac{f(b) - f(a)}{b - a}
\]
Proof. We shall prove the attracting case and leave the repelling case as an exercise.

Let $|f'(p)| < 1$. We must show that $p$ is attracting, i.e. there exists an interval $I = (p - \delta, p + \delta)$ such that if $x \in I$ then

$$\lim_{n \to \infty} f^n(x) = p.$$ 

Since $|f'(p)| < 1$, let us choose some $K$ with $|f'(p)| < K < 1$. But since $f'$ is continuous (i.e. $f$ is $C^1$) we can choose $\delta > 0$ such that if $I = (p - \delta, p + \delta)$ then

$$|f'(x)| < K \quad \text{for all } x \in I.$$

Now by the MVT, for all $x \in \mathbb{R}$

$$f(x) - f(p) = f'(\xi) (x - p)$$
for some \( s \in ](x,p) \text{ if } x > p \)

\[ f(x_0) - f(p) | = |f'(s)| |x_0 - p| \]

\[ < K |x_0 - p| \]

\[ \text{as } K < 1 \]

\[ |x_1 - p| < |x_0 - p| \]

Thus, \( x_1 \in I \).

Applying the MVT again (on \([p,x_1] \) or \([x_1,p] \) ) gives:

\[ |f(x_1) - f(p)| < K |x_1 - p| \]

\[ \text{ie. } |x_2 - p| < K |x_1 - p| \]

\[ < K. K |x_0 - p| \]

\[ \text{ie. } |x_2 - p| < K^2 |x_0 - p| \]
Repeating this argument inductively we deduce

\[ |x_n - p| < K^n |x_0 - p| \]

for all \( n \in \mathbb{N} \)

Taking limits as \( n \to \infty \), and noting that \( 0 < K < 1 \) (so that \( K^n \to 0 \) as \( n \to \infty \)), we get that

\[ \lim_{n \to \infty} |x_n - p| = 0 \]

i.e. \( \lim_{n \to \infty} x_n = p \)

i.e. \( \lim_{n \to \infty} f^n(x_0) = p \).

Therefore \( p \) is attracting.