MTH 4104 **Introduction to Algebra**

The quaternions **Spring 2019**

Complex numbers provide a useful way to turn two-dimensional geometry into algebra. Adding a complex number corresponds to a translation of the complex plane $\mathbb{C} = \mathbb{R}^2$, and multiplying corresponds to scaling and/or rotation.

In the mid-19th century, Irish mathematician William Rowan Hamilton had been searching for a ring that could play the same role with respect to three-dimensional geometry. He discovered his answer in 1843, introducing the ring of *quaternions*.

Hamilton spent the rest of his life on quaternions, but despite this, quaternions turned out to be only a minor sideline in the history of geometry. Around the 1880s, their place in geometry was usurped by *linear algebra* (that you have begun learning in *Vectors and Matrices*) and the *vector calculus* based thereon (that you will learn in *Calculus III*). Nowadays, mathematicians invariably handle rotations using linear algebra, as matrices. Among the advantages of linear algebra is that it works the same way in any number of dimensions, even the non-physical cases of four or more dimensions, unlike the quaternionic approach which is restricted to three. However, quaternions are still used today in some special applications, such as representing rotations in computer graphics and robotics: they take less memory than matrices, and are not susceptible to the problem of *gimbal lock*. An explanation of how to use quaternions for rotations in three dimensions can be found at the link `http://3dgep.com/understanding-quaternions/#Quaternions`.

Of course, this has not diminished the interest in quaternions within algebra. Their importance is illustrated by Theorem 0.3.

Hamilton's idea, after a long time trying without success to make his "3-D" ring out of a set of the form $\{a+b\mathrm{i}+c\mathrm{j} : a,b,c \in \mathbb{R}\}$, was to introduce a fourth coordinate k. The equations came to Hamilton in a flash of insight in the form

$$\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = \mathrm{ijk} = -1,$$

and in this form he cut them into the stones of Broom Bridge in Dublin. There is a plaque on the site today, and it is the focus of occasional mathematical pilgrimages. But these equations are not enough to be a formal definition. Let's examine one.

Definition 0.1. A *quaternion* is a number of the form $\alpha+\beta\mathrm{j}$ where j is a formal symbol and α,β are complex numbers. Addition and multiplication are defined as follows:

$$\begin{aligned}(\alpha+\beta\mathrm{j})+(\gamma+\delta\mathrm{j}) &:= (\alpha+\gamma)+(\beta+\delta)\mathrm{j},\\ (\alpha+\beta\mathrm{j})(\gamma+\delta\mathrm{j}) &:= (\alpha\gamma-\beta\bar{\delta})+(\alpha\delta+\beta\bar{\gamma})\mathrm{j}.\end{aligned}$$

Pay attention to the placement of the bars over δ and γ in the definition. Here $\bar{\delta}$ denotes the *complex conjugate* of δ from Section 2.5 of the main notes. We write

$$\mathbb{H} := \{\alpha+\beta\mathrm{j} : \alpha,\beta \in \mathbb{C}\}$$

for the set of quaternions: $\mathbb{H}$ is for Hamilton. Two quaternions $\alpha+\beta\mathrm{j}$ and $\alpha'+\beta'\mathrm{j}$ are equal precisely when $\alpha=\alpha'$ and $\beta=\beta'$.

Quaternions owe their name to the fact that we need *four* real numbers to uniquely specify a quaternion $q=\alpha+\beta\mathrm{j}$, since if $\alpha=a+b\mathrm{i}$ and $\beta=c+d\mathrm{i}$ for some real numbers a,b,c,d, then

$$q = a+b\mathrm{i}+c\mathrm{j}+d\mathrm{k},$$

where k is a name for ij. Although it hides some of the symmetry of the quaternions, we have set up the definition using two complex numbers instead of four real numbers because there's less work to do in the proofs that way.

Just as $\mathbb{R}$ is a subset of $\mathbb{C}$, we can see $\mathbb{C}$ as a subset of $\mathbb{H}$. That is, we can think of each complex number α as a quaternion $\alpha+0\mathrm{j}$, and this identification respects addition and multiplication.

Theorem 0.2. *$\mathbb{H}$ is a skewfield. In other words, the multiplicative associative, identity and inverse laws, and the distributive law, hold for quaternions.*

We will leave the associative and distributive laws as an exercise, and focus on the inverse law. As before, the multiplicative identity element is $1+0\mathrm{j}$ (i.e. $1\in\mathbb{R}$ when $\mathbb{R}$ is viewed as a subset of $\mathbb{H}$) since

$$1(\alpha+\beta\mathrm{j}) = (1+0\mathrm{j})(\alpha+\beta\mathrm{j}) = (1\alpha-0\bar{\beta})+(1\cdot\beta+0\bar{\alpha})\mathrm{j} = \alpha+\beta\mathrm{j}$$

for any quaternion $\alpha + \beta \mathrm{j} \in \mathbb{H}$.

For the inverse law, let the non-zero quaternion $q = \alpha + \beta \mathrm{j} = a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k}$ be given, and define its *conjugate* by the formula

$$\bar{q} := \bar{\alpha} - \beta \mathrm{j} = a - b\mathrm{i} - c\mathrm{j} - d\mathrm{k}.$$

By the definition of multiplication in $\mathbb{H}$ we compute that

$$q\bar{q} = \bar{q}q = |\alpha|^2 + |\beta|^2.$$

Let $r := q\bar{q}$. Note that r is a *real* number because the modulus $|\alpha|$ of any complex number α is necessarily real. In fact, $r \geq 0$.

Suppose for a contradiction that $r = 0$. Then $|\alpha|^2 + |\beta|^2 = 0$, but the sum of two non-negative real numbers can only be zero if both real numbers are actually zero themselves. Thus $|\alpha|^2 = |\beta|^2 = 0$ which in turn implies that $\alpha = \beta = 0$. We have a reached a contradiction, since $q = \alpha + \beta j$ is a *non-zero* quaternion by assumption.

Therefore r is a non-zero real number, and it is permissible to divide by r. Dividing the equation $q\bar{q} = r$ by r, we obtain

$$q\left(\frac{\bar{q}}{r}\right) = 1 = \left(\frac{\bar{q}}{r}\right)q.$$

So the quaternion $q^{-1} := \frac{a}{r} - \frac{b}{r}\mathrm{i} - \frac{c}{r}\mathrm{j} - \frac{d}{r}\mathrm{k} = \frac{\bar{q}}{r}$ satisfies the equation $qq^{-1} = 1 = q^{-1}q$. □

Note that the Theorem does not say that the set $\mathbb{H}$ of quaternions forms a field. This is because the *commutative law for multiplication* is not satisfied! Indeed, when we multiply j by i we get

$$\mathrm{ji} = (0 + 1\mathrm{j})(\mathrm{i} + 0\mathrm{j}) = (0\mathrm{i} - 1\bar{0}) + (0 \cdot 0 + 1\bar{\mathrm{i}})\mathrm{j} = (-\mathrm{i})\mathrm{j} = -\mathrm{ij}.$$

If Hamilton hadn't introduced a violation of the commutative law here, he would have paid the price in a violation of the inverse law instead. Observe how when we expand the brackets in $(\mathrm{i} - \mathrm{j})(\mathrm{i} + \mathrm{j})$, the cross terms ij and ji do not cancel, preventing the product from working out to 0:

$$(\mathrm{i} - \mathrm{j})(\mathrm{i} + \mathrm{j}) = \mathrm{i}^2 - \mathrm{ji} + \mathrm{ij} - \mathrm{j}^2 = -1 - (-\mathrm{ij}) + \mathrm{ij} - (-1) = 2\mathrm{ij} \neq 0.$$

If $(\mathrm{i} - \mathrm{j})(\mathrm{i} + \mathrm{j})$ **had** been equal to 0, then $(\mathrm{i} - \mathrm{j})$ could not have had any multiplicative inverse x, because we would have

$$\mathrm{i} + \mathrm{j} = 1 \cdot (\mathrm{i} + \mathrm{j}) = x(\mathrm{i} - \mathrm{j})(\mathrm{i} + \mathrm{j}) = x \cdot 0 = 0$$

which would be a contradiction.

We will not prove the following theorem, or even really discuss it, but I feel it important to include it as justification for the quaternions. For the definition of "vector space", refer to the module *Linear Algebra I*.

Theorem 0.3 (Frobenius)**.** *The only skewfields which are also finite-dimensional vector spaces over $\mathbb{R}$ (with the same addition and multiplication) are $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$.*

(In fact the theorem as stated has a pedantic flaw. For example, let's say we constructed a field in the same way as $\mathbb{C}$ except that we used the name j rather than i for the square root of -1. The resulting set, $\{a+b\mathrm{j} : a,b \in \mathbb{R}\}$, is a different set to $\mathbb{C}$. But we want to count it as the same as $\mathbb{C}$ for the purposes of Frobenius' theorem. The concept we need for this is *isomorphism*, which you will encounter in later modules in algebra.)

Tips for computing with quaternions The key equation which allows efficient computation with quaternions, which can be easily checked, is

$$\mathrm{j}z = \bar{z}\mathrm{j} \tag{1}$$

for any complex number z (i.e. quaternion $z+0\mathrm{j}$). In fact, this equation is enough to rediscover the formula

$$(\alpha+\beta\mathrm{j})(\gamma+\delta\mathrm{j}) = (\alpha\gamma-\beta\bar{\delta})+(\alpha\delta+\beta\bar{\gamma})\mathrm{j}$$

that defines quaternion multiplication. To do this, let the quaternions $\alpha+\beta\mathrm{j}$ and $\gamma+\delta\mathrm{j}$ be given. What is their product? We can apply the distributive and associative laws to expand the brackets and get

$$(\alpha+\beta\mathrm{j})(\gamma+\delta\mathrm{j}) = \alpha\gamma+\beta\mathrm{j}\gamma+\alpha\delta\mathrm{j}+\beta\mathrm{j}\delta\mathrm{j}.$$

Now quaternion multiplication is not commutative, so we cannot for example just move the j past the complex number γ in this equation: in general,

$$\beta\mathrm{j}\gamma \neq \beta\gamma\mathrm{j}.$$

However, equation (1) comes to the rescue:

$$\mathrm{j}\gamma = \bar{\gamma}\mathrm{j} \quad \text{and} \quad \mathrm{j}\delta = \bar{\delta}\mathrm{j}.$$

Finally, remembering that $\mathrm{j}^2 = -1$ yields

$$\begin{aligned}(\alpha+\beta\mathrm{j})(\gamma+\delta\mathrm{j}) &= \alpha\gamma+\beta\mathrm{j}\gamma+\alpha\delta\mathrm{j}+\beta\mathrm{j}\delta\mathrm{j}\\ &= \alpha\gamma+\beta\bar{\gamma}\mathrm{j}+\alpha\delta\mathrm{j}+\beta\bar{\delta}\mathrm{j}^2\\ &= (\alpha\gamma-\beta\bar{\delta})+(\alpha\delta+\beta\bar{\gamma})\mathrm{j}.\end{aligned}$$

Bonus link Musical theatre fans will appreciate https://www.youtube.com/watch?v=SZXHoWwBcDc.