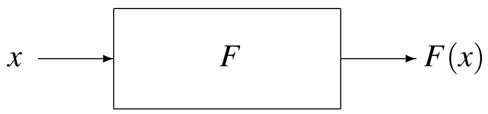
MTH 4104 Introduction to Algebra

Appendix: Functions as relations

You have been using functions for a large part of your mathematical career. In Section 1.2 of the notes, we formalised relations. How can we formalise a function?

Historically, people used to think that a function had to be given by a formula, such as x^2 or $\sin x$. We don't require this any longer. All that is important is that you put in a value for the argument of the function, and out comes a value. Think of a function as a kind of black box:



The name of the function is F. We put x into the black box and $F(x)$ comes out. Be careful not to confuse F, the name written on the black box, with $F(x)$, which is what comes out when x is put in. Sometimes the language makes it hard to keep this straight. For example, there is a function which, when you put in x, outputs x^2. We tend to call this "the function x^2", but it is really "the squaring function", or "the function $x \mapsto x^2$". You see that we have a special symbol $\mapsto$ to denote what the black box does.

To make this formal, we first choose a set X of allowable inputs to the black box. X is called the *domain* of F. Similarly, there will be a set Y which contains all the possible outputs; this is called the *codomain* of F. (We don't necessarily require that every value of Y can come out of the black box. For the squaring function, the domain and the codomain are both equal to $\mathbb{R}$, even though none of the outputs can be negative.)

The important thing is that every input $x \in X$ produces exactly one output $y = F(x) \in Y$. The ordered pair (x, y) is a convenient way of saying that the input x produces the output y. Then we can take the set of all these ordered pairs as a description of the function. Thus we come to the formal definition:

Definition 0.1. Let X and Y be sets. Then a *function* from X to Y is a subset F of $X \times Y$ having the property that, for every $x \in X$, there is exactly one element $y \in Y$

such that $(x,y) \in F$. We write this unique y as $F(x)$. We write $F : X \to Y$ (read "F from X to Y") to mean that F is a function with domain X and codomain Y.

According to this definition, a function $F : X \to X$ with equal domain and codomain is simply a special kind of relation on the set X. Using our notation from Section 3.3, a relation R on X is a function if and only if each set $[x]_R$ has exactly one element. (We would have been able to say the same about functions $F : X \to Y$ with different domain and codomain if we had given a definition for relations between two sets.)

Here is an example. Let $X = Y = \{1,2,3,4,5\}$, and let

$$F = \{(1,1),(2,4),(3,5),(4,4),(5,1)\}.$$

Then F is a function from X to Y, with $F(1) = 1$, $F(2) = 4$, and so on. We can also represent this function in a table.

x	1	2	3	4	5
$F(x)$	1	4	5	4	1

In this particular case, it happens that F is given by a fairly simple formula: $F(x) = -x^2 + 6x - 4$. But a formula was not needed to state the function. The list of elements, or the table, are enough. If we had not been able to come up with a formula, that would not have compromised the legitimacy of F as a function. You should see how this is just like the situation for relations.