**MTH 4104** **Introduction to Algebra**

## On "the natural numbers"

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In the notes, I mention that different mathematicians use the name *natural numbers* for the elements of one of two different sets, $\mathbb{Z}_{\geq 0} = \{0, 1, 2, 3, \ldots\}$ and $\mathbb{Z}_{>0} = \{1, 2, 3, \ldots\}$. My own preference is for the former definition. But Thomas' Calculus textbook uses the latter definition, and it has been decided that our first-year modules should not contradict the textbook in this regard.

Both of these sets play important roles in mathematics.

- $\{0, 1, 2, 3, \ldots\}$ is the set of possible sizes that a finite set can have. In other words, these are the possible answers (other than "infinity") to a "how many" question, which is the original sentiment behind the word "natural". This set also has useful algebraic properties, such as satisfying the *additive identity law* defined later in the notes.
- $\{1, 2, 3, \ldots\}$ is, for example, the set of integers to which the Fundamental Theorem of Arithmetic applies. Positive integers can be factored into prime numbers (including 1, which is the product of zero primes), but there is no way to factor 0.

But perhaps the most frequent invocation of the natural numbers is in a bookkeeping role, to index the terms of a sequence. If we have a sequence, say

$$a_1, a_2, a_3, \ldots,$$

and we want to refer to an arbitrary one of its terms, we can call it $a_i$ where $i$ is a natural number. "Aha", you may say, "that only works if 0 is not a natural number! There is no $a_0$." True, but only because we started counting from 1. If we had labelled our sequence

$$a_0, a_1, a_2, \ldots$$

then we would want the version of the natural numbers with 0 in. The fact that we normally count starting from one is a fact about language and culture, rather than about mathematics proper. So I find this weak as an argument for what "natural number" should mean: we could just use a different indexing practice instead.