Queen Mary University of London

MTH4104 **Introduction to Algebra**

Extra questions for chapter 7 **Spring 2021**

These questions are for you to practice with on your own schedule. You may e.g. want to try some each week as their topics come up in lectures, or use them in exam revision.

7 Groups

7.1 Definition

Question 7.1.1 Let H be the set of all rational numbers of the form a/b where a is an even integer and b is an odd integer. Define an operation $*$ on H by

$$x * y = \frac{x+y}{1-xy}.$$

Prove that $(H,*)$ is a group.

Question 7.1.2 This question shows a more "geometric" way to talk about symmetries and the groups they form than just using permutations.

Let S be the square in the plane $\mathbb{R}^2$ with vertices $(1,1)$, $(1,-1)$, $(-1,1)$, and $(-1,-1)$. Let G be the set of all linear transformations $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $f(S) = S$. (By $f(S)$ I mean the set $\{f(\mathbf{x}) : \mathbf{x} \in S\}$.) These linear transformations are called the *symmetries* of S.

(a) Write out all the elements of G.

(b) Prove that G is a group under the operation of function composition, $(f \circ g)(\mathbf{x}) = f(g(\mathbf{x}))$.

Question 7.1.3 Let $(G,\circ)$ and $(H,*)$ be two groups. Define an operation $\cdot$ on their Cartesian product $G \times H$ by

$$(g,h) \cdot (g',h') = (g \circ g', h * h').$$

Prove that $(G \times H, \cdot)$ is a group.

7.2 Cayley tables

Question 7.2.1 Write down every possible Cayley table of a group whose set of elements is $\{\mathrm{a},\mathrm{b},\mathrm{c}\}$.

7.3 Elementary properties

Question 7.3.1 A group G contains five elements u, v, w, x, y, none of which is the identity element, satisfying

$$uv = w, \qquad vw = x, \qquad wx = y, \qquad xy = u, \qquad yu = v.$$

What is the order of v?

7.4 Units

I have put all the questions about units in the next section.

7.5 The group of units

Question 7.5.1 List all the elements of the multiplicative group $\mathbb{Z}_{15}^{\times}$. Calculate the order of each element.

Question 7.5.2 Write down a Cayley table for the group of units within the ring $M_2(\mathbb{Z}_2)$, i.e. the multiplicative group of invertible 2×2 matrices with coefficients in $\mathbb{Z}_2$.

Question 7.5.3

(a) Let x be a non-negative integer. Prove that

$$2^{x+6k} \equiv_9 2^x$$

for any $k \in \mathbb{N}$.

(b) Using part (a), show that the function $f : \mathbb{Z}_6 \to \mathbb{Z}_9^{\times}$ given by

$$f(X) := [2^x]_9 \qquad \text{where} \quad X = [x]_6 \quad \text{for some} \quad x \in \mathbb{N}$$

is well-defined.

(c) Calculate $f([x]_6)$ for each x in the set $\{0, 1, 2, 3, 4, 5\}$.

(d) Prove that f is a bijection.

(e) Prove that $f(a+b) = f(a)f(b)$ for all a and b in $\mathbb{Z}_6$.

7.6 Subgroups

Question 7.6.1 Let H be the set $\{q^2 : q \in \mathbb{Q}^{\times}\}$. Prove that H is a subgroup of the multiplicative group $\mathbb{Q}^{\times}$.

Question 7.6.2

(a) Let g and h be elements of a group G. Prove that if $gh = hg$, then $g^{-1}h = hg^{-1}$.

(b) Let G be a group, and h an element of G. Let C be the set $\{g \in G : gh = hg\}$. Describe C in words.

(c) Prove that C is a subgroup of G.

Question 7.6.3 Let S be the set of all complex numbers of modulus 1. Prove that S is a subgroup of the multiplicative group $\mathbb{C}^\times$.

7.7 Questions that are really about earlier parts of the module, but use the concept of a group

Question 7.7.1

(a) Let $(G,+)$ be any abelian group. Define a multiplication operation on G by the rule $a \cdot b = 0$ for all $a, b \in G$. Prove that G with the operations $+$ and $\cdot$ is a ring.

(b) Prove that the matrix ring $M_n(G)$ is commutative for all n.