Queen Mary University of London

MTH4104 **Introduction to Algebra**

Extra questions for chapter 6 **Spring 2022**

These questions are for you to practice with on your own schedule. You may e.g. want to try some each week as their topics come up in lectures, or use them in exam revision.

6 Permutations

6.1 Definition and notation

Question 6.1.1 Let m be a positive integer. Let a and b be integers such that $\gcd(a,m) = 1$. Prove that the function f defined by

$$f(x) = [a]_m \cdot x + [b]_m$$

is a permutation of the set $\mathbb{Z}_m$.

6.2 Composition

Question 6.2.1 Let X and Y be finite sets, and $f : X \to Y$ be a function. In lecture I asserted that f has an inverse function if and only if it is a bijection. This is a corollary of the following:

(a) Prove that there exists a function g such that $f \circ g$ is the identity function on Y if and only if f is a surjection.

(b) Prove that there exists a function h such that $h \circ f$ is the identity function on X if and only if f is an injection.

(c) Prove that, if f is a bijection, then the g and h from parts (a) and (b) can be chosen to be equal to each other.

Question 6.2.2 Let R be a relation on the set $\{1,\ldots,n\}$. Let us say that a permutation $f \in S_n$ is a *symmetry* of R if and only if

$$\{(f(x), f(y)) : (x,y) \in R\} = R.$$

Prove that:

(a) The identity permutation e is a symmetry of R.

(b) If f is a symmetry of R, then so is f^{-1}.

(c) If f and g are symmetries of R, then so is $f \circ g$.

6.3 Cycles

Question 6.3.1

(a) Convert the element $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 6 & 1 & 7 & 2 & 3 & 9 & 5 & 8 & 4 \end{pmatrix}$ of S_9 from two-line notation to cycle notation.

(b) Convert the element $g = (1\ 10\ 4\ 6)(5\ 8)(7\ 9\ 3)$ of S_{10} from cycle notation to two-line notation[1].

Question 6.3.2

(a) Let f be a permutation which is a single cycle of length m. Prove that if m is odd, then $f \circ f$ is also a single cycle of length m, and if m is even, then $f \circ f$ decomposes into two disjoint cycles of length $m/2$.

(b) Let $g \in S_n$ be a permutation. Describe a method for answering the following question: does there exist a permutation $f \in S_n$ such that $f \circ f = g$?

[1] If you want more questions of this type, see `http://www.maths.qmul.ac.uk/fink/PermutationComputations.html`.