MTH4104 **Introduction to Algebra**

Extra questions for chapter 5 **Spring 2022**

These questions are for you to practice with on your own schedule. You may e.g. want to try some each week as their topics come up in lectures, or use them in exam revision.

5 Matrices

5.1 Defining matrices

I include in this section questions that are about the addition and multiplication operations on matrices, even if they mention the fact that $\mathrm{M}_n(R)$ is a ring.

Question 5.1.1

(a) Write down a *general* element of the ring $\mathrm{M}_2(\mathrm{M}_2(\mathbb{R}))$. (That is, write out the structure of an element, with brackets etc. in the necessary places, using distinct variables to stand in for the real numbers.)

(b) Give the formula for multiplying two elements of $\mathrm{M}_2(\mathrm{M}_2(\mathbb{R}))$.

(c) How is $\mathrm{M}_2(\mathrm{M}_2(\mathbb{R}))$ related to $\mathrm{M}_4(\mathbb{R})$?

Solution (a) If you have read the notes carefully, you'll have noticed that this is done there. An element of $M_2(M_2(\mathbb{R}))$ is a 2×2 matrix of 2×2 matrices of real numbers. Such a construct looks like this:

$$\begin{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} & \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \\ \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} & \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix} \end{pmatrix}$$

where the a_{ij}, b_{ij}, c_{ij} and d_{ij} all lie in $\mathbb{R}$.

(b) Let M be the element written down in part (a). For convenience we'll give names to the matrices which are its entries, $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$, and so on, so that

$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Let N be another element of $M_2(M_2(\mathbb{R}))$, and write it $N = \begin{pmatrix} E & F \\ G & H \end{pmatrix}$, where E, F, G, H are 2×2 real matrices. Then

$$MN = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} E & F \\ G & H \end{pmatrix} = \begin{pmatrix} AE+BG & AF+BH \\ CE+DG & CF+DH \end{pmatrix},$$

which leaves us eight products and four sums to do in $M_2(\mathbb{R})$. The top-left entry is

$$\begin{aligned} AE+BG &= \begin{pmatrix} a_{11}e_{11}+a_{12}e_{21} & a_{11}e_{12}+a_{12}e_{22} \\ a_{21}e_{11}+a_{22}e_{21} & a_{21}e_{12}+a_{22}e_{22} \end{pmatrix} + \begin{pmatrix} b_{11}g_{11}+b_{12}g_{21} & b_{11}g_{12}+b_{12}g_{22} \\ b_{21}g_{11}+b_{22}g_{21} & b_{21}g_{12}+b_{22}g_{22} \end{pmatrix} \\ &= \begin{pmatrix} a_{11}e_{11}+a_{12}e_{21}+b_{11}g_{11}+b_{12}g_{21} & a_{11}e_{12}+a_{12}e_{22}+b_{11}g_{12}+b_{12}g_{22} \\ a_{21}e_{11}+a_{22}e_{21}+b_{21}g_{11}+b_{22}g_{21} & a_{21}e_{12}+a_{22}e_{22}+b_{21}g_{12}+b_{22}g_{22} \end{pmatrix} \end{aligned}$$

and one has formulae for the other three entries that look the same except for changing the letters. All in all, the product is

$$\begin{pmatrix} \begin{pmatrix} a_{11}e_{11}+a_{12}e_{21}+b_{11}g_{11}+b_{12}g_{21} & a_{11}e_{12}+a_{12}e_{22}+b_{11}g_{12}+b_{12}g_{22} \\ a_{21}e_{11}+a_{22}e_{21}+b_{21}g_{11}+b_{22}g_{21} & a_{21}e_{12}+a_{22}e_{22}+b_{21}g_{12}+b_{22}g_{22} \end{pmatrix} & \begin{pmatrix} a_{11}f_{11}+a_{12}f_{21}+b_{11}h_{11}+b_{12}h_{21} & a_{11}f_{12}+a_{12}f_{22}+b_{11}h_{12}+b_{12}h_{22} \\ a_{21}f_{11}+a_{22}f_{21}+b_{21}h_{11}+b_{22}h_{21} & a_{21}f_{12}+a_{22}f_{22}+b_{21}h_{12}+b_{22}h_{22} \end{pmatrix} \\ \begin{pmatrix} c_{11}e_{11}+c_{12}e_{21}+d_{11}g_{11}+d_{12}g_{21} & c_{11}e_{12}+c_{12}e_{22}+d_{11}g_{12}+d_{12}g_{22} \\ c_{21}e_{11}+c_{22}e_{21}+d_{21}g_{11}+d_{22}g_{21} & c_{21}e_{12}+c_{22}e_{22}+d_{21}g_{12}+d_{22}g_{22} \end{pmatrix} & \begin{pmatrix} c_{11}f_{11}+c_{12}f_{21}+d_{11}h_{11}+d_{12}h_{21} & c_{11}f_{12}+c_{12}f_{22}+d_{11}h_{12}+d_{12}h_{22} \\ c_{21}f_{11}+c_{22}f_{21}+d_{21}h_{11}+d_{22}h_{21} & c_{21}f_{12}+c_{22}f_{22}+d_{21}h_{12}+d_{22}h_{22} \end{pmatrix} \end{pmatrix}$$

(c) You may have noticed that, if you rub away the internal parentheses, a 2×2 matrix of 2×2 matrices looks the same as a 4×4 matrix. But that's not all: the arithmetic on both kinds of object works the same way! If you compute the product of the two matrices

$$\begin{pmatrix} a_{11} & a_{12} & b_{11} & b_{12} \\ a_{21} & a_{22} & b_{21} & b_{22} \\ c_{11} & c_{12} & d_{11} & d_{12} \\ c_{21} & c_{22} & d_{21} & d_{22} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} e_{11} & e_{12} & f_{11} & f_{12} \\ e_{21} & e_{22} & f_{21} & f_{22} \\ g_{11} & g_{12} & h_{11} & h_{12} \\ g_{21} & g_{22} & h_{21} & h_{22} \end{pmatrix}$$

in $M_4(\mathbb{R})$, which are M and N without their internal parentheses, the result you get will be exactly the answer to (b) above without its internal parentheses. The similarity is more obvious for sums: in either $M_2(M_2(\mathbb{R}))$ or $M_4(\mathbb{R})$, to sum two matrices you sum corresponding real-number elements.

Therefore $M_2(M_2(\mathbb{R}))$ and $M_4(\mathbb{R})$ are really "the same ring" in different notations: there's just a bunch of extra pointless brackets written in the former ring. To use the more technical language that will be taught in subsequent modules in algebra. we say that the rings $M_2(M_2(\mathbb{R}))$ and $M_4(\mathbb{R})$ are *isomorphic*.

Also, nothing about this question used properties of the real numbers other than the ring axioms. So the proof above will also show that $M_2(M_2(R))$ and $M_4(R)$ are "the same" for any ring R.

5.2 Matrix rings

Question 5.2.1 Find a multiplicative inverse for the matrix

$$\begin{pmatrix} [7]_{13} & [6]_{13} \\ [10]_{13} & [4]_{13} \end{pmatrix}$$

within $\mathrm{M}_2(\mathbb{Z}_{13})$.

Solution Since $\mathbb{Z}_{13}$ is a field, we may use the familiar procedure for inversion of 2×2 matrices. The rule is that the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

has as its inverse the matrix

$$\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

supposing that this latter matrix is defined, that is as long as $ad-bc\neq 0$. But it is important that $\mathbb{Z}_{13}$ be a **field** for this approach to be correct: it doesn't work for just any ring! Proving it uses not only the multiplicative inverse law, to divide by $ad-bc$, but also (less obviously) the commutative law for multiplication.

In the present instance, we compute that

$$([7]_{13}[4]_{13}-[6]_{13}[10]_{13})^{-1}=[-32]_{13}^{-1}=[7]_{13}^{-1}=[2]_{13}$$

The field $\mathbb{Z}_{13}$ is small enough that brute force is probably the quickest way to observe that $[7]_{13}^{-1}=[2]_{13}$, but you could also have used the Euclidean algorithm. Next, the inverse matrix is

$$\begin{aligned}
&([7]_{13}[4]_{13}-[6]_{13}[10]_{13})^{-1}\begin{pmatrix} [4]_{13} & -[6]_{13} \\ -[10]_{13} & [7]_{13} \end{pmatrix}\\
&=[2]_{13}\begin{pmatrix} [4]_{13} & [7]_{13} \\ [3]_{13} & [7]_{13} \end{pmatrix}\\
&=\begin{pmatrix} [8]_{13} & [1]_{13} \\ [6]_{13} & [1]_{13} \end{pmatrix}.
\end{aligned}$$

If you were unsure about the applicability of the above procedure, the results could of course be checked, by working out the products

$$\begin{pmatrix} [7]_{13} & [6]_{13} \\ [10]_{13} & [4]_{13} \end{pmatrix}\begin{pmatrix} [8]_{13} & [1]_{13} \\ [6]_{13} & [1]_{13} \end{pmatrix}=\begin{pmatrix} [7\cdot 8+6\cdot 6]_{13} & [7\cdot 1+6\cdot 1]_{13} \\ [10\cdot 8+4\cdot 6]_{13} & [10\cdot 1+4\cdot 1]_{13} \end{pmatrix}=\begin{pmatrix} [1]_{13} & [0]_{13} \\ [0]_{13} & [1]_{13} \end{pmatrix}$$

and

$$\begin{pmatrix} [8]_{13} & [1]_{13} \\ [6]_{13} & [1]_{13} \end{pmatrix}\begin{pmatrix} [7]_{13} & [6]_{13} \\ [10]_{13} & [4]_{13} \end{pmatrix}=\begin{pmatrix} [8\cdot 7+1\cdot 10]_{13} & [8\cdot 6+1\cdot 4]_{13} \\ [6\cdot 7+1\cdot 10]_{13} & [6\cdot 6+1\cdot 4]_{13} \end{pmatrix}=\begin{pmatrix} [1]_{13} & [0]_{13} \\ [0]_{13} & [1]_{13} \end{pmatrix}.$$

What would you do if you didn't know, or (sensibly) didn't trust, the determinant rule for computing the inverse? In this case, the task would be to solve for elements $w,x,y,z\in\mathbb{Z}_{13}$ that bring about

$$\begin{pmatrix} [7]_{13} & [6]_{13} \\ [10]_{13} & [4]_{13} \end{pmatrix}\begin{pmatrix} w & x \\ y & z \end{pmatrix}=\begin{pmatrix} [1]_{13} & [0]_{13} \\ [0]_{13} & [1]_{13} \end{pmatrix}$$

and

$$\begin{pmatrix} w & x \\ y & z \end{pmatrix}\begin{pmatrix} [7]_{13} & [6]_{13} \\ [10]_{13} & [4]_{13} \end{pmatrix}=\begin{pmatrix} [1]_{13} & [0]_{13} \\ [0]_{13} & [1]_{13} \end{pmatrix}.$$

Performing either one of these multiplications and equating entries gives you two lots of two linear equations in two unknowns. If we use the first, we get

$$\begin{pmatrix} [7]_{13}w+[6]_{13}y & [7]_{13}x+[6]_{13}z \\ [10]_{13}w+[4]_{13}y & [10]_{13}x+[4]_{13}z \end{pmatrix}=\begin{pmatrix} [1]_{13} & [0]_{13} \\ [0]_{13} & [1]_{13} \end{pmatrix}$$

which implies

$$\begin{aligned}
[7]_{13}\, w + [6]_{13}\, y &= [1]_{13} \\
[10]_{13}\, w + [4]_{13}\, y &= [0]_{13}; \\
[7]_{13}\, x + [6]_{13}\, z &= [0]_{13} \\
[10]_{13}\, x + [4]_{13}\, z &= [1]_{13}.
\end{aligned}$$

These can be solved by the usual isolate-and-substitute procedure, and produce the same answer as above.

Question 5.2.2 Suppose R is a nontrivial ring with identity.

(a) Specify a non-zero 2×2 matrix N with coefficients in R such that $N^2 = 0$.

(b) Using part (a) or otherwise, prove that $\mathrm{M}_2(R)$ does not satisfy the multiplicative inverse law.

Solution (a) Clearly if we take the zero matrix $O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, then

$$O^2 = OO = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 \cdot 0 + 0 \cdot 0 & 0 \cdot 0 + 0 \cdot 0 \\ 0 \cdot 0 + 0 \cdot 0 & 0 \cdot 0 + 0 \cdot 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = O.$$

But we need a *non-zero* matrix. So let's try changing just one of the zeros in O to a non-zero entry, let's say the multiplicative identity $1 \in R$, which we know is not zero because R satisfies the nontriviality law.

The matrix $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ doesn't work since it is its own square (check this for yourself!). For similar reasons, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ doesn't work either. However

$$N := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

does work, because

$$N^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 \cdot 0 + 1 \cdot 0 & 0 \cdot 1 + 1 \cdot 0 \\ 0 \cdot 0 + 0 \cdot 0 & 0 \cdot 1 + 0 \cdot 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = O.$$

(b) We know that $\mathrm{M}_2(R)$ is a ring from the lectures. Hence we may apply our general argument about nonzero elements $x = N$, $y = N$ in $\mathrm{M}_2(R)$ that satisfy $xy = 0$ inside $\mathrm{M}_2(R)$, to deduce that $\mathrm{M}_2(R)$ is not a field. To wit, if x had a multiplicative inverse x^{-1}, then

$$y = 1 \cdot y = x^{-1}xy = x^{-1} \cdot 0 = 0$$

which is false.

Question 5.2.3 Let R be a ring. Let $\alpha, \beta, \gamma, \delta \in R$ and let $A = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$ and $B = \begin{pmatrix} \gamma & \delta \\ -\delta & \gamma \end{pmatrix}$.

(a) Calculate AB and $A+B$.

(b) Write down a function $\psi : \mathbb{C} \to \mathrm{M}_2(\mathbb{R})$ that satisfies the rules

$$\psi(zw) = \psi(z)\psi(w) \quad \text{and} \quad \psi(z+w) = \psi(z) + \psi(w).$$

(c) Suppose you knew that $\mathrm{M}_2(\mathbb{R})$ was a ring but were unfamiliar with $\mathbb{C}$. Could you use part (b) to help prove that $\mathbb{C}$ is a field?

Solution (a) Applying the definition of matrix multiplication gives

$$\begin{aligned} AB &= \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \begin{pmatrix} \gamma & \delta \\ -\delta & \gamma \end{pmatrix} = \\ &= \begin{pmatrix} \alpha\gamma + \beta(-\delta) & \alpha\delta + \beta\gamma \\ (-\beta)\gamma + \alpha(-\delta) & (-\beta)\delta + \alpha\gamma \end{pmatrix} \end{aligned}$$

and

$$A + B = \begin{pmatrix} \alpha+\gamma & \beta+\delta \\ -\beta-\delta & \alpha+\gamma \end{pmatrix}.$$

(b) Because multiplication in $\mathbb{C}$ is commutative, we can rewrite AB as follows in the case that R is $\mathbb{C}$:

$$AB = \begin{pmatrix} \alpha\gamma - \beta\delta & \alpha\delta + \beta\gamma \\ -(\alpha\delta + \beta\gamma) & \alpha\gamma - \beta\delta \end{pmatrix}.$$

Note that this matrix is *in the same form* as A and B, in that

$$AB = \begin{pmatrix} \varepsilon & \zeta \\ -\zeta & \varepsilon \end{pmatrix}$$

where $\varepsilon = \alpha\gamma - \beta\delta$ and $\zeta = \alpha\delta + \beta\gamma$.

Having this in mind, if we define the function

$$\psi : \mathbb{C} \to M_2(\mathbb{R})$$

by the rule

$$\psi(\alpha + \beta\mathrm{i}) = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix},$$

where $\alpha, \beta \in \mathbb{R}$, then part (a) proves that

$$\psi(zw) = \psi(z)\psi(w) \quad \text{and} \quad \psi(z+w) = \psi(z) + \psi(w)$$

for all complex numbers $z = \alpha + \beta\mathrm{i}$ and $w = \gamma + \delta\mathrm{i}$.

In technical terms, what we have shown is that the image of ψ is a *subring* of $\mathrm{M}_2(\mathbb{R})$, i.e. a ring that is a subset of $\mathrm{M}_2(\mathbb{R})$ with the same addition and multiplication operations.

(c) The computations above can be used to save some effort when proving that $\mathbb{C}$ is a field (though whether this is a good tradeoff of effort is a matter of taste...) The two associative laws and the commutative and distributive laws for addition in $\mathbb{C}$ follow automatically from the corresponding laws for matrices, because matrices form a ring.

Similarly, if you knew how to invert matrices with coefficients in $\mathbb{R}$, you could use matrix inversion to figure out what multiplicative inverses in $\mathbb{C}$ must be and use that to prove the multiplicative inverse law, without having to know how to divide complex numbers going in.

Question 5.2.4 The point of this question is to illustrate that the rule for inverting matrices using the determinant only works over *commutative* rings. It uses the ring $\mathbb{H}$ of quaternions, so you'll need to refer to the supplementary notes on quaternions on QMPlus.

(a) Compute the square of the matrix of quaternions

$$A = \begin{pmatrix} i & j \\ j & i \end{pmatrix}$$

within $\mathrm{M}_2(\mathbb{H})$. Bear in mind that $\mathbb{H}$ is a noncommutative ring: be careful that you don't accidentally swap the order of factors when multiplying.

(b) Does A have a multiplicative inverse? Why or why not?

(c) Let us denote the entries of A by $a_{i,j}$. Compute the quaternion

$$a_{1,1}a_{2,2} - a_{1,2}a_{2,1}.$$

Solution (a) We compute

$$A^2 = \begin{pmatrix} i & j \\ j & i \end{pmatrix}\begin{pmatrix} i & j \\ j & i \end{pmatrix} = \begin{pmatrix} i\cdot i + j\cdot j & i\cdot j + j\cdot i \\ j\cdot i + i\cdot j & j\cdot j + i\cdot i \end{pmatrix} = \begin{pmatrix} -1-1 & k-k \\ -k+k & -1-1 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}.$$

(b) Yes, A has a multiplicative inverse. It is

$$A^{-1} = \begin{pmatrix} -i/2 & -j/2 \\ -j/2 & -i/2 \end{pmatrix}.$$

You may check that this is the inverse of A by working out both products AA^{-1} and $A^{-1}A$ and observing that you get $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, the multiplicative identity matrix in $M_2(\mathbb{H})$.

How did I find this matrix A^{-1}? The key fact I wanted you to notice is that the product in part (a) is a real scalar multiple of the identity matrix. Since multiplication of a quaternion by a real number is commutative, we have the equality $A \cdot -\frac{1}{2} = -\frac{1}{2} \cdot A$ of matrix-scalar products. Now the equation $A^2 = -2I$ implies

$$A \cdot -\frac{1}{2}A = -\frac{1}{2}A \cdot A = -\frac{1}{2} \cdot -2I = I,$$

implying that $-\frac{1}{2}A$ is the multiplicative inverse of A. And indeed, the matrix A^{-1} above is just $-\frac{1}{2}$ times A.

(c) We compute

$$a_{1,1}a_{2,2} - a_{1,2}a_{2,1} = i \cdot i - j \cdot j = (-1) - (-1) = 0.$$

Therefore we have an example illustrating that the determinantal formula does not work over the quaternions: the determinant of A comes out as 0, but nonetheless A does have an inverse.